Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 1, Number 1, March 2019, 93–105 DOI:10.12150/jnma.2019.93

The Complete Biorthogonal Expansion Theorem and Its Application to a Class of Rectangular Plate Equations*

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Abstract In this paper, we first establish the separable *Hamiltonian* system of rectangular cantilever thin plate bending problems by choosing proper dual vectors. Then using the characteristics of off-diagonal infinite-dimensional *Hamiltonian* operator matrix, we derive the biorthogonal relationships of the eigenfunction systems and based on it we further obtain the complete biorthogonal expansion theorem. Finally, applying this theorem we obtain the general solutions of rectangular cantilever thin plate bending problems with two opposite edges slidingly supported.

Keywords Rectangular cantilever thin plate, Hamiltonian operator, biorthogonal expansion theorem, general solutions, completeness.

MSC(2010) 46E30, 47A70, 47B39, 74B05.

1. Introduction

Rectangular thin plates are important structural components that are used in various engineering applications such as plates in rigid pavements of highways, bridge and houses decks and traffic zones of airports. Bending analysis of rectangular thin plates is one of the challenging issues in theory and engineering, especially for rectangular cantilever thin plates which is an important structural element. Actually its bending has been one of the most difficult problems in the theory of elastic thin plate due to the complexity in both the governing equation and the boundary conditions.

In this paper, we consider the bending problems of cantilever thin plates in the rectangular region $\Omega = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq b\}$. The governing equations of the plates are

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \tag{1.1}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0, \tag{1.2}$$

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*The authors were supported by NSF of China (Nos. 11671142 and 11771075) and Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0, \tag{1.3}$$

with

$$M_x = -D(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2}), \tag{1.4}$$

$$M_y = -D(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2}), \tag{1.5}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 W}{\partial x \partial y}, \tag{1.6}$$

where W is the transverse deflection of plate midplane, D is the flexural rigidity, q is the distributed transverse load, M_x, M_y and M_{xy} are the bending moments and the torsional moment, respectively. The internal forces of the plate are

$$Q_x = -D\frac{\partial(\nabla^2 W)}{\partial x}, \tag{1.7}$$

$$Q_y = -D\frac{\partial(\nabla^2 W)}{\partial y}, \tag{1.8}$$

and

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}, \tag{1.9}$$

$$V_y = Q_y + \frac{\partial M_{xy}}{\partial x}, \tag{1.10}$$

where Q_x, Q_y, V_x and V_y are shear forces and total shear forces, respectively.

In reference [6], the basic equations for rectangular thin plate were transferred to a *Hamiltonian* canonical equation and the symplectic superposition method was applied to obtain the exact bending solutions. In the end, two numerical examples were provided to illustrate the accuracy of the proposed method. However, the completeness of the eigenfunction systems of the corresponding *Hamiltonian* operator has not been established. In [1] the authors discussed and obtained the completeness of the eigenfunction systems of the *Hamiltonian* operator stated in the rectangular plates with two opposite edges slidingly supported through the symplectic eigenfunction expansion approach. This approach was originated by W. Zhong and X. Zhong [15] to solve a class of eigenvalue problems of non-self-adjoint operators in mathematical physics and it has been applied to various branches of mechanics and engineering sciences, see [2,3,7,8,13,14,16] among others.

In recent years, Luo etc [9,10] further studied a new systematic methodology for theory of elasticity and found that the symplectic orthogonality relationships could be decomposed into two symmetrical and independent sub-orthogonality relationships for the orthotropic plane elasticity and thin plate theory. And the biorthogonal relationships of elasticity was also extended into three-dimensional couple stress problems (see [11]). The biorthogonal relationships not only includes but also is simpler than the symplectic orthogonality relationship. Therefore, the method of biorthogonal expansion in solving the elastic mechanics equations has obvious advantages in calculation than the symplectic eigenfunction expansion method, which makes the calculation more concise. By utilizing this method Hou etc [5] investigated the eigenfunction systems of the *Hamiltonian* operator for the Mindlin plate and obtained the exact solutions to deflections and bending moments for the Mindlin plate with fully simply supported sides.

In this article, making use of a complete biorthogonal expansion theorem we devote to looking for the general solutions of the equations (1.1)-(1.3) for the bending problems of cantilever thin plates. We first study the eigenfunction systems of the infinite-dimensional *Hamiltonian* operator associated to the considered equations. Then we utilize the complete biorthogonal expansion method to establish the completeness of the eigenfunction systems of the corresponding *Hamiltonian* operator. Finally the general solutions of the equations (1.1)-(1.3) are achieved by applying the theorem and, meanwhile, the feasibility of using complete biorthogonal expansion method is obtained as well. It is seen that our obtained results are more advantageous than those in [6].

2. Preliminaries

In this section we recall some definitions and lemmas, which will be used in our discussion.

Definition 2.1 ([4]). Let X be a Hilbert space, $H = \begin{bmatrix} A & F \\ G & -A^* \end{bmatrix} : \mathcal{D}(H) \subset X \times X \to X \times X$ be a densely defined closed linear operator, where A is a densely defined and closed linear operator in X, and F and G are (*symmetric*) self-adjoint operators. Then, H is called an infinite-dimensional Hamiltonian operator. Specially, if $A = 0$, H is called an off-diagonal infinite-dimensional Hamiltonian operator, and the following evolution equation

$$\frac{\partial U(t,x)}{\partial t} = HU + f,$$

is called the infinite-dimensional separable Hamiltonian system, Herein, f denotes the vector of external force.

Definition 2.2 ([4]). Let T be a linear operator in space X with the eigenvalue λ. If there exists a non-zero element $u^0 \in D(T)$ such that $Tu^0 = \lambda u^0$, and u^0 is called the basic eigenfunction of T. Further, if there exists $u^1 \in D(T)$ such that

$$Tu^1 = \lambda u^1 + u^0, \quad u^1 \neq 0,$$

then u^1 is called the $(first - order)$ Jordan form eigenfunction of T. By induction, if the Jordan form eigenfunction u^{k-1} of order $k-1$ of T has been defined, then the Jordan form eigenfunction u^k of order k is defined by the formula

$$Tu^k = \lambda u^k + u^{k-1}, \quad u^k \neq 0.$$

Definition 2.3. Let X be a Hilbert space, the vector set $\{u_k\}_{k=-\infty}^{+\infty}$ in X is called to be complete in the sense of Cauchy principal value, if for any $x \in X$ there exist constant sequences $\{c_k\}_{k=0}^{+\infty}$ and $\{c_{-k}\}_{k=1}^{+\infty}$ such that

$$x = c_0 u_0 + \sum_{k=1}^{\infty} (c_k u_k + c_{-k} u_{-k}).$$

Definition 2.4. If a linear operator T in space X has Jordan eigenfunctions, then the functions composed of the basic eigenfunctions and Jordan eigenfunctions of T are called the generalized eigenfunction systems of T.

Definition 2.5. Let $X = L^2[0,a] \times L^2[0,a]$, we define a new inner product in the *Hilbert* space X as follows:

$$\langle u_1, u_2 \rangle = \int_0^a u_1(x)^T u_2(x) \mathrm{d}x, \ \ for \ any \ u_1, u_2 \in X.$$

Lemma 2.1 ([12]). *The following orthogonal set of functions*

$$\left\{ \cos \frac{k\pi}{a} \mid k = 0, 1, 2, \cdots \right\}$$

is complete in the Hilbert space $L^2[0,a]$ with the standard inner product. Thus, the corresponding Fourier series of any $g(x) \in L^2[0,a]$ converges to $g(x)$ in $L^2[0,a]$, i.e.

$$g(x) = \frac{1}{a} \int_0^a g(\xi) \mathrm{d}\xi + \sum_{m=1}^{+\infty} \left[\frac{2}{a} \int_0^a g(\xi) \cos \frac{k\pi}{a} \xi \mathrm{d}\xi \right] \cos \frac{k\pi}{a} x.$$

3. Separable Hamiltonian system

In this section, we will transform the governing equation of the rectangular cantilever thin plate into an infinite-dimensional separable *Hamiltonian* system by introducing the state parameters. On the rectangular region Ω, the boundary conditions for a plate slidingly supported at $x = 0$ and $x = a$ are

$$\frac{\partial W(x,y)}{\partial x} = V_x(x,y) = 0, \quad \text{for} \ \ x = 0, a. \tag{3.1}$$

In order to obtain the separable Hamiltonian system of rectangular plates, according to Eqs. (1.9), (1.10) and (1.3) , we have

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + q = 0, \tag{3.2}$$

Setting

$$\frac{\partial W}{\partial y} = \theta. \tag{3.3}$$

From Eq. (1.5), we can get

$$\frac{\partial \theta}{\partial y} = -\nu \frac{\partial^2 W}{\partial x^2} - \frac{1}{D} M_y. \tag{3.4}$$

In view of Eq. (1.6), we have

$$M_{xy} = -D(1-\nu) \frac{\partial \theta}{\partial x}. \tag{3.5}$$

Combining Eqs. (1.5), (1.6), (1.7), (1.9) and (3.2), we find

$$\frac{\partial V_y}{\partial y} = D(1-\nu^2)\frac{\partial^4 W}{\partial x^4} - \nu \frac{\partial^2 M_y}{\partial x^2} - q. \tag{3.6}$$

Similarly, from Eqs. (1.2), (1.10) and (3.5), we obtain that

$$\frac{\partial M_y}{\partial y} = V_y + 2D(1-\nu)\frac{\partial^2 \theta}{\partial x^2}. \tag{3.7}$$

Now, let the $Hilbert$ space $X = L^2(0,a)$ and $Z = X \times X$, and we define on $Z \times Z$ and Z the operators

$$H = \begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2D(1-\nu)\frac{\partial^2}{\partial x^2} & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{D} & -\nu\frac{\partial^2}{\partial x^2} \\ -\nu\frac{\partial^2}{\partial x^2} & -D(1-\nu^2)\frac{\partial^4}{\partial x^4} \end{bmatrix},$$

where the domain of operator B and C are respectively given by

$$\mathcal{D}(B) = \left\{ \begin{bmatrix} \theta \\ T \end{bmatrix} \in Z \,\middle|\, \theta', \theta'' \in X \text{ and } \theta' \text{ are absolutely continuous} \right\},$$

and

$$\mathcal{D}(C) = \left\{ \begin{bmatrix} -M_y \\ W \end{bmatrix} \in Z \,\middle|\, \begin{array}{c} W'(0) = W'(a) = W'''(0) = W'''(a) = 0, \\ W', M_y', W'', M_y'', W''', W^{(4)} \in X \text{ and } M_y', \\ W', W'', W''' \text{ are absolutely continuous} \end{array} \right\}.$$

Then, putting $V_y = -T$ and defining the full state vectors $U = (\theta, T, -M_y, W)^T$, $f = (0, q, 0, 0)^T$, we rewrite Eqs. (3.3), (3.4), (3.6) and (3.7) as the following form

$$\frac{\partial U}{\partial y} = HU + f. \tag{3.8}$$

It was prove in [6] that H is a $Hamiltonian$ operator. Therefore, the considered rectangular cantilever thin plate bending problems are now transformed into the separable $Hamiltonian$ system (3.8).

4. Complete biorthogonal expansion theorem

In this section, using the biorthogonal relationships of the eigenfunction systems , we establish a complete biorthogonal expansion theorem for the bending problems of rectangular cantilever plate.

For conciseness, we denote $-M_y$ by M. In order to the obtain biorthogonal relationships, we set

$$U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, U_1 = \begin{bmatrix} \theta \\ T \end{bmatrix}, U_2 = \begin{bmatrix} M \\ W \end{bmatrix}, f_1 = \begin{bmatrix} 0 \\ q \end{bmatrix}, f_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.1}$$

Then we can rewrite Eq. (3.8) as

$$\dot{U}_1 = CU_2 + f_1, \tag{4.2}$$

$$\dot{U}_2 = BU_1 + f_2. \tag{4.3}$$

Differentiate Eq. (4.3) about y, and substitute into Eq. (4.2) yield that

$$\ddot{U}_2 = BCU_2 + Bf_1. \tag{4.4}$$

Put $F = BC$, $\check{f} = Bf_1$, then Eq. (4.4) becomes

$$\ddot{U}_2 = FU_2 + \check{f}, \tag{4.5}$$

where

$$F = \begin{bmatrix} (\nu-2)\frac{\partial^2}{\partial x^2} & -D(1-\nu)^2\frac{\partial^4}{\partial x^4} \\ \frac{1}{D} & -\nu\frac{\partial^2}{\partial x^2} \end{bmatrix}, \quad \check{f} = \begin{bmatrix} q \\ 0 \end{bmatrix}.$$

According to boundary conditions (3.1), we obtain the domain of the operator F as

$$\mathcal{D}(F) = \left\{ \begin{bmatrix} M \\ W \end{bmatrix} \in Z \left| \begin{array}{c} W'(0) = W'(a) = W'''(0) = W'''(a) = 0, \\ W', M', W'', M'', W''', W^{(4)} \in X \text{ and} \\ M', W', W'', W''' \text{ are absolutely continuous} \end{array} \right. \right\}. \tag{4.6}$$

If we can solve U_2 from Eq. (4.5), then it's easy to get U_1 through Eq. (4.3). By straightforward calculation, we find that the operator F has the zero eigenvalue and nonzero eigenvalues $\lambda_m = (\frac{m\pi}{a})^2$, $m = 1, 2, \cdots$. Moreover, the corresponding basic eigenfunctions of the nonzero eigenvalues $\lambda_m = (\frac{m\pi}{a})^2$ are

$$u_m^0(x) = \begin{bmatrix} D(\frac{m\pi}{a})^2(1-\nu)\cos\frac{m\pi}{a}x \\ \cos\frac{m\pi}{a}x \end{bmatrix}, \ m = 1, 2, \cdots,$$

and the first-order Jordan eigenfunctions of the nonzero eigenvalue $\lambda_m = (\frac{m\pi}{a})^2$ are given by

$$u_m^1(x) = \begin{bmatrix} D[(\frac{m\pi}{a})^2(1-\nu)+1]\cos\frac{m\pi}{a}x \\ \cos\frac{m\pi}{a}x \end{bmatrix}, \ m = 1, 2, \cdots.$$

In addition, the basic eigenfunctions and first-order Jordan eigenfunctions of the zero eigenvalue are respectively

$$v_0^0(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad v_0^1(x) = \begin{bmatrix} D \\ 1 \end{bmatrix}.$$

Furthermore, by careful calculation, we can get the following proposition on their biorthogonal relationships.

Proposition 4.1. *The eigenfunction systems* $\{u_m^0(x)\}_{m=1}^{+\infty}, \{u_m^1(x)\}_{m=1}^{+\infty}, v_0^0(x)$ *and* $v_0^1(x)$ *of the operator* F *have the following biorthogonal relationships:*

$$\langle u_m^0, v_0^0\rangle = 0\ ,\ \langle u_m^1, v_0^0\rangle = 0\ ,\ \langle v_0^0, u_n^1\rangle = 0\ , \langle v_0^0, u_n^0\rangle = 0\ ,\ \langle v_0^0, v_0^0\rangle = \langle v_0^0, v_0^1\rangle = a$$

$$\langle u_m^0, v_0^1\rangle = 0\ ,\ \langle u_m^1, v_0^1\rangle = 0\ ,\ \langle v_0^1, u_n^1\rangle = 0\ , \langle v_0^1, u_n^0\rangle = 0\ ,\ \langle v_0^1, v_0^1\rangle = (D^2+1)a$$

$$\langle v_0^1, v_0^0\rangle = a\ , m,n = 1,2,\cdots$$

and

$$\langle u_m^1, u_n^1\rangle = \begin{cases} 0 & m \neq n, \\ \frac{a[D^2(\lambda_m - \lambda_m\nu+1)^2+1]}{2} & m = n, \end{cases} \quad m,n = 1,2,\cdots,$$

$$\langle u_m^0, u_n^0\rangle = \begin{cases} 0 & m \neq n, \\ \frac{a[D^2\lambda_m^2(1-\nu)^2+1]}{2} & m = n, \end{cases} \quad m,n = 1,2,\cdots,$$

$$\langle u_m^1, u_n^0\rangle = \langle u_m^0, u_n^1\rangle = \begin{cases} 0 & m \neq n, \\ \frac{a[D^2\lambda_m(1-\nu)[\lambda_m(1-\nu)+1]+1]}{2} & m = n, \end{cases} \quad m,n = 1,2,\cdots,$$

where inner product $\langle u_1, u_2\rangle$ *is defined in Definition 2.5.*

Proof. The eigenfunction systems of the the operator F are respectively

$$u_m^0(x) = \begin{bmatrix} D(\frac{m\pi}{a})^2(1-\nu)\cos\frac{m\pi}{a}x \\ \cos\frac{m\pi}{a}x \end{bmatrix}, u_m^1(x) = \begin{bmatrix} D[(\frac{m\pi}{a})^2(1-\nu)+1]\cos\frac{m\pi}{a}x \\ \cos\frac{m\pi}{a}x \end{bmatrix},$$

$m = 1,2,\cdots,$

$$u_n^0(x) = \begin{bmatrix} D(\frac{n\pi}{a})^2(1-\nu)\cos\frac{n\pi}{a}x \\ \cos\frac{n\pi}{a}x \end{bmatrix}, u_n^1(x) = \begin{bmatrix} D[(\frac{n\pi}{a})^2(1-\nu)+1]\cos\frac{n\pi}{a}x \\ \cos\frac{n\pi}{a}x \end{bmatrix},$$

$n = 1,2,\cdots.$ and

$$v_0^0(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_0^1(x) = \begin{bmatrix} D \\ 1 \end{bmatrix}.$$

By direct calculation, it follows that

$$\langle u_m^0, v_0^0\rangle = \int_0^a 0\cdot D(\frac{m\pi}{a})^2(1-\nu)\cos\frac{m\pi}{a}xdx + \int_0^a 1\cdot\cos\frac{m\pi}{a}xdx = 0,$$

$$\langle u_m^1, v_0^0\rangle = \int_0^a 0\cdot D[(\frac{m\pi}{a})^2(1-\nu)+1]\cos\frac{m\pi}{a}xdx + \int_0^a 1\cdot\cos\frac{m\pi}{a}xdx = 0,$$

$$\langle v_0^1, v_0^1\rangle = \int_0^a D\cdot Ddx + \int_0^a 1\cdot 1dx = (D^2+1)a,$$

in a similar way

$$\langle v_0^0, u_n^1\rangle = 0\ , \langle v_0^0, u_n^0\rangle = 0\ ,\ \langle v_0^0, v_0^0\rangle = \langle v_0^0, v_0^1\rangle = a\ ,\ \langle u_m^0, v_0^1\rangle = 0$$

$$\langle u_m^1, v_0^1\rangle = 0\ ,\ \langle v_0^1, u_n^1\rangle = 0\ , \langle v_0^1, u_n^0\rangle = 0\ ,\ \langle v_0^1, v_0^0\rangle = a\ , m, n = 1, 2, \cdots .$$

Then

$$\begin{aligned}\langle u_m^1, u_n^1\rangle &= \int_0^a D[(\frac{m\pi}{a})^2(1-\nu)+1]\cos\frac{m\pi}{a}x\cdot D[(\frac{n\pi}{a})^2(1-\nu)+1]\cos\frac{n\pi}{a}xdx \\ &\quad + \int_0^a \cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{a}xdx, \\ &= \begin{cases} 0 & m\neq n, \\ \frac{a[D^2(\lambda_m-\lambda_m\nu+1)^2+1]}{2} & m=n, \end{cases} \quad m,n=1,2,\cdots,\end{aligned}$$

$$\begin{aligned}\langle u_m^0, u_n^1\rangle &= \int_0^a D(\frac{m\pi}{a})^2(1-\nu)\cos\frac{m\pi}{a}x\cdot D[(\frac{n\pi}{a})^2(1-\nu)+1]\cos\frac{n\pi}{a}xdx \\ &\quad + \int_0^a \cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{a}xdx, \\ &= \begin{cases} 0 & m\neq n, \\ \frac{a[D^2\lambda_m(1-\nu)[\lambda_m(1-\nu)+1]+1]}{2} & m=n, \end{cases} \quad m,n=1,2,\cdots,\end{aligned}$$

similarly

$$\langle u_m^0, u_n^0\rangle = \begin{cases} 0 & m\neq n, \\ \frac{a[D^2\lambda_m^2(1-\nu)^2+1]}{2} & m=n, \end{cases} \quad m,n=1,2,\cdots,$$

$$\langle u_m^1, u_n^0\rangle = \begin{cases} 0 & m\neq n, \\ \frac{a[D^2\lambda_m(1-\nu)[\lambda_m(1-\nu)+1]+1]}{2} & m=n, \end{cases} \quad m,n=1,2,\cdots,$$

where $\lambda_m = (\frac{m\pi}{a})^2,\ m = 1, 2, \cdots$. The proof is completed.

□

Due to Proposition 4.1, we then have the following complete biorthogonal expansion theorem for the rectangular cantilever plate bending problem.

Theorem 4.1. *The generalized eigenfunction systems $v_0^0(x), v_0^1(x)$, $\{u_m^0(x)\}_{m=1}^{+\infty}$ and $\{u_m^1(x)\}_{m=1}^{+\infty}$ of the operator F are complete in the sense of Cauchy principal value in Z .*

Proof. We need to prove that value in Z. for any $G(x) = [g_1(x), g_2(x)]^T \in Z$, there exist constant sequences $d_0^0,\ d_0^1,\ \{c_m^0\}_{m=1}^{+\infty}$ and $\{c_m^1\}_{m=1}^{+\infty}$ such that

$$G(x) = d_0^0 v_0^0 + d_0^1 v_0^1 + \sum_{m=1}^{+\infty}\left(c_m^0 u_m^0 + c_m^1 u_m^1\right).$$

In fact, due to Proposition 4.1 we have

$$\begin{cases} \langle G, v_0^1\rangle = d_0^0\langle v_0^0, v_0^1\rangle + d_0^1\langle v_0^1, v_0^1\rangle, \\ \langle G, v_0^0\rangle = d_0^0\langle v_0^0, v_0^0\rangle + d_0^1\langle v_0^1, v_0^0\rangle, \end{cases} \tag{4.7}$$

and

$$\begin{cases} \langle G, u_m^1 \rangle = c_m^0 \langle u_m^0, u_m^1 \rangle + c_m^1 \langle u_m^1, u_m^1 \rangle, \\ \langle G, u_m^0 \rangle = c_m^0 \langle u_m^0, u_m^0 \rangle + c_m^1 \langle u_m^1, u_m^0 \rangle, \end{cases} \tag{4.8}$$

in which

$$d_0^0 = -\frac{1}{aD}\int_0^a g_1(\xi)\mathrm{d}\xi + \frac{1}{a}\int_0^a g_2(\xi)\mathrm{d}\xi, \quad d_0^1 = \frac{1}{aD}\int_0^a g_1(\xi)\mathrm{d}\xi,$$

$$c_m^0 = -\frac{2}{a}\left(\frac{1}{D}\int_0^a g_1(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi + (\lambda_m\nu - \lambda_m - 1)\int_0^a g_2(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi\right),$$

$$c_m^1 = \frac{2}{a}\left(\frac{1}{D}\int_0^a g_1(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi - (\lambda_m - \lambda_m\nu)\int_0^a g_2(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi\right).$$

Therefore,

$$\begin{aligned} G =& d_0^0 v_0^0 + d_0^1 v_0^1 + \sum_{m=1}^{+\infty}\left(c_m^0 u_m^0 + c_m^1 u_m^1\right) \\ =& \begin{bmatrix} \frac{1}{a}\int_0^a g_1(\xi)\mathrm{d}\xi \\ \frac{1}{a}\int_0^a g_2(\xi)\mathrm{d}\xi \end{bmatrix} + \sum_{m=1}^{+\infty}\begin{bmatrix} \left(\frac{2}{a}\int_0^a g_1(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi\right)\cos\frac{m\pi}{a}x \\ \left(\frac{2}{a}\int_0^a g_2(\xi)\cos\frac{m\pi}{a}\xi\mathrm{d}\xi\right)\cos\frac{m\pi}{a}x \end{bmatrix}. \end{aligned}$$

So in view of Lemma 2.1, the generalized eigenfunction systems of operator F are complete in Hilbert space Z in the sense of Cauchy principal value. This proof is completed. □

5. General solutions for rectangular cantilever plate bending problems

Now, we are on the position to apply Theorem 4.1 to find the general solutions of Eq. (3.8). For this, we first obtain the general solutions of Eq. (4.5).

From the superposition principle of solutions and Theorem 4.1, the general solution of Eq. (4.5) has the following form:

$$U_2(x,y) = \begin{bmatrix} M \\ W \end{bmatrix} = l_0^0(y)v_0^0(x) + l_0^1(y)v_0^1(x) + \sum_{m=1}^{+\infty}\left(t_m^0(y)u_m^0(x) + t_m^1(y)u_m^1(x)\right). \tag{5.1}$$

The non-homogeneous term $\check{f}$ is expanded as:

$$\check{f}(x,y) = \begin{bmatrix} q \\ 0 \end{bmatrix} = k_0^0(y)v_0^0(x) + k_0^1(y)v_0^1(x) + \sum_{m=1}^{+\infty}\left(p_m^0(y)u_m^0(x) + p_m^1(y)u_m^1(x)\right). \tag{5.2}$$

By virtue of Proposition 4.1, we see that

$$\begin{cases} \langle \check{f}, v_0^1 \rangle = k_0^0 \langle v_0^0, v_0^1 \rangle + k_0^1 \langle v_0^1, v_0^1 \rangle, \\ \langle \check{f}, v_0^0 \rangle = k_0^0 \langle v_0^0, v_0^0 \rangle + k_0^1 \langle v_0^1, v_0^0 \rangle, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \langle \check{f}, u_m^1 \rangle = p_m^0 \langle u_m^0, u_m^1 \rangle + p_m^1 \langle u_m^1, u_m^1 \rangle, \\ \langle \check{f}, u_m^0 \rangle = p_m^0 \langle u_m^0, u_m^0 \rangle + p_m^1 \langle u_m^1, u_m^0 \rangle. \end{cases} \tag{5.4}$$

By straightforward calculation, from Eqs. (5.3) and (5.4), the coefficients of $\check{f}(x, y)$ can be obtained as

$$k_0^0(y) = -\frac{1}{aD} \int_0^a q(\xi, y) \mathrm{d}\xi, \; k_0^1(y) = \frac{1}{aD} \int_0^a q(\xi, y) \mathrm{d}\xi \tag{5.5}$$

$$p_m^0(y) = -\frac{2}{aD} \int_0^a q(\xi, y) \cos \frac{m\pi}{a} \xi \mathrm{d}\xi, \; p_m^1(y) = \frac{2}{aD} \int_0^a q(\xi, y) \cos \frac{m\pi}{a} \xi \mathrm{d}\xi. \tag{5.6}$$

Substituting (5.1) and (5.2) into Eq. (4.5), we then get

$$\begin{cases} \ddot{t_m^0} = \lambda_m t_m^0 + t_m^1 + p_m^0, \\ \ddot{t_m^1} = \lambda_m t_m^1 + p_m^1, \\ \ddot{l_0^0} = l_0^1 + k_0^0, \\ \ddot{l_0^1} = k_0^1, \end{cases}$$

from which we obtain that

$$\begin{aligned} l_0^0(y) &= b_{1m}^0 - \int y\beta(y) \mathrm{d}y + \left(b_{2m}^0 + \int \beta(y) \mathrm{d}y \right) y, \\ l_0^1(y) &= b_{1m}^1 - \int y\alpha(y) \mathrm{d}y + \left(b_{2m}^1 + \int \alpha(y) \mathrm{d}y \right) y, \\ t_m^0(y) &= c_{1m}^0 e^{\frac{m\pi}{a} y} + c_{2m}^0 e^{-\frac{m\pi}{a} y} + \frac{a}{2m\pi} \left(e^{\frac{m\pi}{a} y} \int \varphi(y) e^{-\frac{m\pi}{a} y} \mathrm{d}y - e^{-\frac{m\pi}{a} y} \int \varphi(y) e^{\frac{m\pi}{a} y} \mathrm{d}y \right), \\ t_m^1(y) &= c_{1m}^1 e^{\frac{m\pi}{a} y} + c_{2m}^1 e^{-\frac{m\pi}{a} y} + \frac{a}{2m\pi} \left(e^{\frac{m\pi}{a} y} \int f(y) e^{-\frac{m\pi}{a} y} \mathrm{d}y - e^{-\frac{m\pi}{a} y} \int f(y) e^{\frac{m\pi}{a} y} \mathrm{d}y \right), \end{aligned}$$

where

$$\begin{aligned} \alpha(y) &= k_0^1(y) = \frac{1}{aD} \int_0^a q(\xi, y) \mathrm{d}\xi, \\ \beta(y) &= b_{1m}^1 - \int y\alpha(y) \mathrm{d}y + \left(b_{2m}^1 + \int \alpha(y) \mathrm{d}y \right) y - \alpha(y), \\ f(y) &= p_m^1(y) = \frac{2}{aD} \int_0^a q(\xi, y) \cos \frac{m\pi}{a} \xi \mathrm{d}\xi, \\ \varphi(y) &= c_{1m}^1 e^{\frac{m\pi}{a} y} + c_{2m}^1 e^{-\frac{m\pi}{a} y} + \frac{a}{2m\pi} \left(e^{\frac{m\pi}{a} y} \int f(y) e^{-\frac{m\pi}{a} y} \mathrm{d}y - e^{-\frac{m\pi}{a} y} \int f(y) e^{\frac{m\pi}{a} y} \mathrm{d}y \right) - f(y). \end{aligned}$$

Now, substituting $l_0^0(y)$, $l_0^1(y)$, $t_m^0(y)$ and $t_m^1(y)$ into Eq. (5.1) gives that

$$
\begin{aligned}
M(x,y) = &\left(b_{1m}^1 + b_{2m}^1 y + y\int \alpha(y)\mathrm{d}y - \int y\alpha(y)\mathrm{d}y\right) D \\
&+ \sum_{m=1}^{+\infty}\left(\left(c_{1m}^0 + \frac{a}{2m\pi}\int \varphi(y)e^{-\frac{m\pi}{a}y}\mathrm{d}y\right)e^{\frac{m\pi}{a}y}\right. \\
&\left.+\left(c_{2m}^0 - \frac{a}{2m\pi}\int \varphi(y)e^{\frac{m\pi}{a}y}\mathrm{d}y\right)e^{-\frac{m\pi}{a}y}\right) \\
&D\left(\frac{m\pi}{a}\right)^2(1-\nu)\cos\frac{m\pi}{a}x \\
&+\left(\left(c_{1m}^1 + \frac{a}{2m\pi}\int f(y)e^{-\frac{m\pi}{a}y}\mathrm{d}y\right)e^{\frac{m\pi}{a}y}\right. \\
&\left.+\left(c_{2m}^1 - \frac{a}{2m\pi}\int f(y)e^{\frac{m\pi}{a}y}\mathrm{d}y\right)e^{-\frac{m\pi}{a}y}\right) \\
&D\left[\left(\frac{m\pi}{a}\right)^2(1-\nu)+1\right]\cos\frac{m\pi}{a}x,
\end{aligned}
$$

and

$$
\begin{aligned}
W(x,y) =& b_{1m}^0 + b_{1m}^1 + (b_{2m}^0 + b_{2m}^1)y \\
&+\left(\int(\alpha(y)+\beta(y))\mathrm{d}y\right)y - \left(\int(\alpha(y)+\beta(y))y\mathrm{d}y\right) \\
&+\sum_{m=1}^{+\infty}\left(\left(c_{1m}^0 + c_{1m}^1 + \frac{a}{2m\pi}\int(\varphi(y)+f(y))e^{-\frac{m\pi}{a}y}\mathrm{d}y\right)e^{\frac{m\pi}{a}y}\right. \\
&\left.+\left(c_{2m}^0 + c_{2m}^1 - \frac{a}{2m\pi}\int(\varphi(y)+f(y))e^{\frac{m\pi}{a}y}\mathrm{d}y\right)e^{-\frac{m\pi}{a}y}\right)\cos\frac{m\pi}{a}x.
\end{aligned}
$$

Because B is reversible, from the second formula of Eq. (4.3) it follows that

$$
U_1 = B^{-1}\dot{U}_2. \tag{5.7}
$$

Thus by Eq. (5.7), we find that

$$
\begin{aligned}
\theta(x,y) =& b_{2m}^0 + b_{2m}^1 + \int(\alpha(y)+\beta(y))\mathrm{d}y \\
&+\sum_{m=1}^{+\infty}\left(\left(\frac{1}{2}\int(\varphi(y)+f(y))e^{-\frac{m\pi}{a}y}\mathrm{d}y + \frac{m\pi(c_{1m}^0 + c_{1m}^1)}{a}\right)e^{\frac{m\pi}{a}y}\right. \\
&\left.+\left(\frac{1}{2}\int(\varphi(y)+f(y))e^{\frac{m\pi}{a}y}\mathrm{d}y - \frac{m\pi(c_{2m}^0 + c_{2m}^1)}{a}\right)e^{-\frac{m\pi}{a}y}\right)\cos\frac{m\pi}{a}x.
\end{aligned}
$$

$$T(x,y)=\left(b_{2m}^1+\int\alpha(y)\mathrm{d}y\right)D+\sum_{m=1}^{+\infty}\left(\left(\frac{1}{2}\int\varphi(y)e^{-\frac{m\pi}{a}y}\mathrm{d}y+\frac{m\pi}{a}c_{1m}^0\right)e^{\frac{m\pi}{a}y}\right.$$
$$\left.+\left(\frac{1}{2}\int\varphi(y)e^{\frac{m\pi}{a}y}\mathrm{d}y-\frac{m\pi}{a}c_{2m}^0\right)e^{-\frac{m\pi}{a}y}\right)D\left(\frac{m\pi}{a}\right)^2(\nu-1)\cos\frac{m\pi}{a}x$$
$$+\left(\left(\frac{1}{2}\int f(y)e^{-\frac{m\pi}{a}y}\mathrm{d}y+\frac{m\pi}{a}c_{1m}^1\right)e^{\frac{m\pi}{a}y}\right.$$
$$\left.+\left(\frac{1}{2}\int f(y)e^{\frac{m\pi}{a}y}\mathrm{d}y-\frac{m\pi}{a}c_{2m}^1\right)e^{-\frac{m\pi}{a}y}\right)$$
$$D\left[\left(\frac{m\pi}{a}\right)^2(\nu-1)+1\right]\cos\frac{m\pi}{a}x,$$

where the unknown coefficients $b_{im}^0, b_{im}^1, c_{im}^0$ *and* $c_{im}^1 (i=1,2; m=1,2,\cdots)$ are determined by the boundary conditions at the remaining two edges.

To sum up, applying Theorem 4.1, we obtain the general solution of the rectangular cantilever plate bending problems.

6. Conclusion

The separable *Hamiltonion* system of rectangular cantilever thin plate bending problem is studied in this paper. By the single side product of two symmetric operator matrices, the eigenvalues and eigenfunctions of the corresponding *Hamiltonion* operator are obtained. Then, the biorthogonal relationships and completeness of the eigenfunction systems are derived by taking full advantage of the structural characteristics of the off-diagonal *Hamiltonian* operator matrix. It is seen that, the biorthogonal relationships of the eigenfunctions makes the computation easier and more convenient, and its completeness guarantees the convergence of the series of the solution function for complete biorthogonal expansion. Finally, the general solutions of the bending problems of the rectangular cantilever thin plate achieved successfully through the complete biorthogonal expansion theorem. The obtained results of this paper can also be applied to discuss many other mechanics problems of multivariable isotropic.

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