

# *N*-dark Soliton Solutions for the Multi-component Maccari System\*

Yi Zhang<sup>1,†</sup>, Chujun Chen<sup>1,†</sup>, Yinkang Xu<sup>1,†</sup> and Yinjie Lu<sup>1,†</sup>

**Abstract** By virtue of the KP hierarchy reduction method, the *N*-dark soliton solutions of the multi-component Maccari system are constructed. Taking the coupled Maccari system for instance, the *N*-dark soliton solutions are further obtained in terms of determinants. In addition, in contrast with bright-bright soliton collisions, the dynamical analysis shows that the collisions of dark-dark solitons are elastic and there is no energy exchange among solitons in different components. What's more, we also investigate the dark-dark soliton bound states including stationary and moving ones.

**Keywords** *N*-dark soliton solutions, the multi-component Maccari system, KP hierarchy reduction method.

**MSC(2010)** 35Q55, 35C08, 81Q05.

## 1. Introduction

During the past decades, solitons in coupled nonlinear Schrödinger (CNLS) type equations have been studied intensively due to their intriguing interest and their applied realms. Many efforts have been made on coupled systems which describe the interaction of short wave packets with long waves in nonlinear dispersive media, as they are frequently used in the fields of nonlinear optics, fluid dynamics and others. Depending on the relative sign between the group velocity dispersion/diffraction and nonlinearity, solitons have two distinct types, namely, bright and dark solitons. The appearance of multi-component CNLS equations as dynamical equations in various areas of physics leads to the identification of bright-, dark-, bright-dark-, and dark-bright-type solitons. In the case of focusing Manakov system, it has been reported that there exhibit certain novel inelastic collision properties, which has not been observed in scalar counterpart. In addition, in the defocusing Manakov system, two bright-dark solitons can form a stationary bound state, in other words, solitons undergo elastic collision without shape change in this case. All these interesting interaction behaviors can be described by multi-soliton solutions in the underlying integrable system. To our knowledge, results are still scarce for the study on bright-dark and dark-dark types soliton propagation and their collision dynamics.

---

<sup>†</sup>the corresponding author.

Email address: zy2836@163.com(Y. Zhang), 1257317388@qq.com(C. Chen), 547861764@qq.com(Y. Xu), 904117651@qq.com(Y. Lu)

<sup>1</sup>Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

\*The authors were supported by the National Natural Science Foundation of China (11371326).

Being motivated by the above reasons, in present work, we consider the Maccari system [16, 23, 26, 30]

$$i\Phi_t + \Phi_{xx} - u\Phi = 0, \quad (1.1)$$

$$u_y = (\sigma\Phi\Phi^*)_x, \quad (1.2)$$

where  $\sigma = \pm 1$ ,  $\Phi$  is complex while  $u$  is real; the asterisk represents complex conjugate hereafter. And this system was introduced by Maccari that usually used to describe the motion of localized isolated waves in physics. Its two-component generalization is given by

$$i\Phi_t^{(1)} + \Phi_{xx}^{(1)} - u\Phi^{(1)} = 0, \quad (1.3)$$

$$i\Phi_t^{(2)} + \Phi_{xx}^{(2)} - u\Phi^{(2)} = 0, \quad (1.4)$$

$$u_y = (\sigma_1\Phi^{(1)}\Phi^{(1)*} + \sigma_2\Phi^{(2)}\Phi^{(2)*})_x, \quad (1.5)$$

where  $\sigma_1, \sigma_2 = \pm 1$ .

For the Maccari system (1.1)-(1.2), many studies have been done. Uthayakumar etc investigated its integrability property by means of the singularity structure analysis [20]. In addition, Lai and Chow have obtained its two-dromion solutions based on the coalescence of wavenumbers technique [13]. By virtue of the variable separation approach, Zhang etc constructed many coherent soliton structures such as dromions, foldon and solitoff [27, 28]. In recent work, its various rational solutions have been presented by He etc utilizing the Hirota's bilinear method [24]. And the general mixed  $N$ -soliton solutions of the multi-component Maccari system have been investigated by Han and Chen [9]. However, the general dark-dark  $N$ -soliton solutions of the multi-component Maccari system have not been exhibited so far.

It should be pointed out that the KP reduction technique is an effective way to derive soliton solutions of integrable systems, which was firstly investigated by the Kyoto school [11, 17]. So far it has been applied to construct soliton solutions in many equations such as the mKdV equation, the NLS equation and others. The reduction of Toda lattice hierarchy with constrained KP systems to derive dark and bright solitons has been established respectively [21, 22]. Additionally, by means of this reduction technique, Ohta and Yang have obtained the general  $N$ -dark-dark soliton solutions in the CNLS equations [18]. Also based on this method, the general bright-dark  $N$ -soliton solutions have already been found for the CNLS equations [7], the YO system [1] and the Mel'nikov system. Moreover, the KP reduction technique has also been applied to derive rational solutions including rogue waves solutions of integrable equations, see also the literatures [2, 14, 19, 25, 29].

The paper is organized. In Sec.2, by means of the KP reduction technique, we construct the general  $N$ -dark-dark soliton solutions of the two-component Maccari system. Besides, the dynamics of one and two solitons are discussed in Sec.3. In Sec.4, we derive the dark-dark soliton bound states, which contain the stationary and the moving ones. In Sec.5, the general  $N$ -dark-dark soliton solutions of the multi-component Maccari system in Gram determinant form are presented.

## 2. $N$ -dark-dark soliton solutions of the two-component Maccari system

In this part, we give the dark-dark  $N$ -soliton solutions of the coupled Maccari system.

Through the variable transformation

$$\Phi^{(1)} = \rho_1 e^{i\theta_1} \frac{g}{f}, \Phi^{(2)} = \rho_2 e^{i\theta_2} \frac{h}{f}, u = -2(\log f)_{xx}, \quad (2.1)$$

where  $f$  is real,  $g$  and  $h$  are complex, meanwhile,  $\theta_i = \alpha_i x - \alpha_i^2 t$ ,  $\alpha_i$  and  $\rho_i$  are two real constants, the two-component Maccari system (1.3)-(1.5) can be transformed into the bilinear equations

$$(D_x^2 + 2i\alpha_1 D_x + iD_t)g \cdot f = 0, \quad (2.2)$$

$$(D_x^2 + 2i\alpha_2 D_x + iD_t)h \cdot f = 0, \quad (2.3)$$

$$D_x D_y f \cdot f = \sigma_1 \rho_1^2 (f^2 - gg^*) + \sigma_2 \rho_2^2 (f^2 - hh^*), \quad (2.4)$$

where the operator  $D$  is defined by [8]

$$D_x^l D_y^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, y, t) \cdot g(x', y', t')|_{x'=x, y'=y, t'=t}. \quad (2.5)$$

**Theorem 2.1.** *The  $N$ -dark-dark soliton solutions for the coupled Maccari system (1.3)-(1.5) are*

$$\Phi^{(1)} = \rho_1 e^{i(\alpha_1 x - \alpha_1^2 t)} \frac{g}{f}, \Phi^{(2)} = \rho_2 e^{i(\alpha_2 x - \alpha_2^2 t)} \frac{h}{f}, u = -2(\log f)_{xx}, \quad (2.6)$$

where

$$\begin{aligned} f &= \left| \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N \times N}, \\ g &= \left| \delta_{ij} + \left( \frac{p_i - i\alpha_1}{p_j^* + i\alpha_1} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N \times N}, \\ h &= \left| \delta_{ij} + \left( \frac{p_i - i\alpha_2}{p_j^* + i\alpha_2} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N \times N}, \end{aligned} \quad (2.7)$$

with

$$\xi_j = p_j x + \frac{1}{2} \left( \frac{\sigma_1 \rho_1^2}{p_j - i\alpha_1} + \frac{\sigma_2 \rho_2^2}{p_j - i\alpha_2} \right) y + ip_j^2 t + \xi_{j0},$$

where  $p_j$  and  $\xi_{j0}$  are complex constants.

In what follows, we proceed to show how the dark-dark solitons are derived with the help of the KP reduction technique. The proof of this theorem is based on the following elementary lemma.

**Lemma 2.1** (Lemma 1, [11]). *The following bilinear equations in the KP hierarchy:*

$$(D_{x_1}^2 - D_{x_2} + 2aD_{x_1})\tau(k+1, l) \cdot \tau(k, l) = 0, \quad (2.8)$$

$$(D_{x_1}^2 - D_{x_2} + 2bD_{x_1})\tau(k, l+1) \cdot \tau(k, l) = 0, \quad (2.9)$$

$$(D_{x_1}D_{x_{-1}} - 2)\tau(k, l) \cdot \tau(k, l) = -2\tau(k+1, l)\tau(k-1, l), \quad (2.10)$$

$$(D_{x_1}D_{y_{-1}} - 2)\tau(k, l) \cdot \tau(k, l) = -2\tau(k, l+1)\tau(k, l-1), \quad (2.11)$$

where  $a$  and  $b$  are complex constants, and  $k$  and  $l$  are integers, have the Gram type determinant solutions

$$\tau(k, l) = |m_{ij}(k, l)|_{1 \leq i, j \leq N}, \quad (2.12)$$

where the entries of the determinant are given by

$$m_{ij}(k, l) = c_{ij} + \int \varphi_i(k, l)\psi_j(k, l)dx_1, \quad (2.13)$$

$$\varphi_i(k, l) = (p_i - a)^k (p_i - b)^l e^{\theta_i}, \quad (2.14)$$

$$\psi_j(k, l) = \left(-\frac{1}{q_j + a}\right)^k \left(-\frac{1}{q_j + b}\right)^l e^{\tilde{\theta}_j}, \quad (2.15)$$

with

$$\theta_i = \frac{1}{p_i - a}x_{-1} + \frac{1}{p_i - b}y_{-1} + p_i x_1 + p_i^2 x_2 + \theta_{i0}, \quad (2.16)$$

$$\tilde{\theta}_j = \frac{1}{q_j + a}x_{-1} + \frac{1}{q_j + b}y_{-1} + q_j x_1 - q_j^2 x_2 + \tilde{\theta}_{j0}, \quad (2.17)$$

where  $c_{ij}, p_i, q_j, \theta_{i0}, \tilde{\theta}_{j0}, (i, j = 1, 2, \dots, N)$  are complex constants.

**Proof.** If one assumes  $x_{-1}, y_{-1}, x_1$  are real,  $x_2, a(=i\alpha_1), b(=i\alpha_2)$  are pure imaginary and  $q_i = p_i^*, \tilde{\theta}_{j0} = \theta_{j0}^*, c_{ji} = c_{ij}^* = \delta_{ij}$  ( $\delta_{ij}$  is the Kronecker symbol), then we can get

$$\tilde{\theta}_j = \theta_j^*, m_{ji}(k, l) = m_{ij}^*(-k, -l), \tau(k, l) = \tau^*(-k, -l). \quad (2.18)$$

Moreover, we let

$$f = \tau(0, 0), g = \tau(1, 0), h = \tau(0, 1), g^* = \tau(-1, 0), h^* = \tau(0, -1), \quad (2.19)$$

hence, the bilinear equations (2.8)-(2.11) become

$$(D_{x_1}^2 - D_{x_2} + 2i\alpha_1 D_{x_1})g \cdot f = 0, \quad (2.20)$$

$$(D_{x_1}^2 - D_{x_2} + 2i\alpha_2 D_{x_1})h \cdot f = 0, \tag{2.21}$$

$$(D_{x_1} D_{x_{-1}} - 2)f \cdot f = -2gg^*, \tag{2.22}$$

$$(D_{x_1} D_{y_{-1}} - 2)f \cdot f = -2hh^*. \tag{2.23}$$

Furthermore, by considering the independent variables changes

$$x_1 = x, x_2 = it, x_{-1} = \frac{1}{2}\sigma_1\rho_1^2 y, y_{-1} = \frac{1}{2}\sigma_2\rho_2^2 y, \tag{2.24}$$

the bilinear equations (2.20)-(2.23) are transformed into the bilinear form (2.2)-(2.4). Thus, putting the transformation (2.24) into the  $f, g, h$  in (2.19), we can immediately derive an alternative form for solutions of Eqs.(1.3)-(1.5) in Theorem 2.1. So above all, these complete the proof of the theorem.  $\square$

### 3. The dynamical analysis of the dark-dark soliton solutions

In this section, we discuss the dynamics of dark-dark soliton solutions in Eqs.(1.3)-(1.5) by applying the above theorem.

#### 3.1. One-soliton solution

Taking  $N = 1$  in the Eqs.(2.6)-(2.7), we can exhibit the single soliton solution. Thus, the determinants read

$$f = 1 + \frac{1}{p_1 + p_1^*} e^{\xi_1 + \xi_1^*}, \tag{3.1}$$

$$g = 1 - \frac{1}{p_1 + p_1^*} \frac{p_1 - i\alpha_1}{p_1^* + i\alpha_1} e^{\xi_1 + \xi_1^*}, \tag{3.2}$$

$$h = 1 - \frac{1}{p_1 + p_1^*} \frac{p_1 - i\alpha_2}{p_1^* + i\alpha_2} e^{\xi_1 + \xi_1^*}, \tag{3.3}$$

and the single dark-dark soliton solution reads

$$\Phi^{(1)} = \frac{\rho_1}{2} e^{i(\alpha_1 x - \alpha_1^2 t)} \times \left[ 1 + K_1^{(1)} + (K_1^{(1)} - 1) \tan h \left( \frac{\xi_1 + \xi_1^* + \Theta_1}{2} \right) \right], \tag{3.4}$$

$$\Phi^{(2)} = \frac{\rho_2}{2} e^{i(\alpha_2 x - \alpha_2^2 t)} \times \left[ 1 + K_1^{(2)} + (K_1^{(2)} - 1) \tan h \left( \frac{\xi_1 + \xi_1^* + \Theta_1}{2} \right) \right], \tag{3.5}$$

$$u = \frac{1}{2} (p_1 + p_1^*)^2 \sec h^2 \left( \frac{\xi_1 + \xi_1^* + \Theta_1}{2} \right), \tag{3.6}$$

with

$$e^{\Theta_1} = \frac{1}{p_1 + p_1^*} = \frac{1}{2a_1},$$

$$K_1^{(1)} = -\frac{p_1 - i\alpha_1}{p_1^* + i\alpha_1} = -\frac{a_1 + i(b_1 - \alpha_1)}{a_1 - i(b_1 - \alpha_1)},$$

$$K_1^{(2)} = -\frac{p_1 - i\alpha_2}{p_1^* + i\alpha_2} = -\frac{a_1 + i(b_1 - \alpha_2)}{a_1 - i(b_1 - \alpha_2)},$$

$$\xi_1 + \xi_1^* = 2a_1x + \left[ \frac{\sigma_1\rho_1^2 a_1}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{\sigma_2\rho_2^2 a_1}{a_1^2 + (b_1 - \alpha_2)^2} \right] y - 4a_1b_1t + 2\xi_{10R},$$

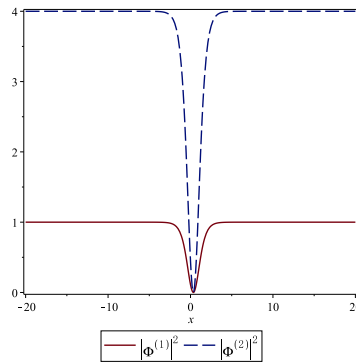
where  $p_1 = a_1 + ib_1$ , and  $a_1, b_1, \xi_{10R}$  are real constants.

From (3.4)-(3.6), the intensity functions of  $|\Phi^{(1)}|, |\Phi^{(2)}|$  and  $u$  of these solitons move at velocity  $2b_1$  along the  $x$ -direction and  $\frac{4b_1}{\frac{\sigma_1\rho_1^2}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{\sigma_2\rho_2^2}{a_1^2 + (b_1 - \alpha_2)^2}}$  along the  $y$ -direction respectively. As  $x, y \rightarrow \pm\infty$ ,  $|\Phi^{(1)}| \rightarrow |\rho_1|, |\Phi^{(2)}| \rightarrow |\rho_2|, -u \rightarrow 0$ .

Denoting  $K_1^{(1)} = \exp(2i\phi_1^{(1)})$  and  $K_1^{(2)} = \exp(2i\phi_1^{(2)})$ , when  $x$  and  $y$  vary from  $-\infty$  to  $+\infty$ , it should be pointed out that the  $\Phi^{(1)}$  and  $\Phi^{(2)}$  components phase shifts are in the range of  $2\phi_1^{(1)}$  and  $2\phi_1^{(2)}$  while the phase of the  $-u$  component is zero, if  $2\phi_1^{(1)}$  and  $2\phi_1^{(2)}$  are the phases of  $K_1^{(1)}$  and  $K_1^{(2)}$ . Without loss of generality, we can restrict  $-\pi < 2\phi_1^{(1)}, 2\phi_1^{(2)} \leq \pi$ , ( $-\frac{\pi}{2} < \phi_1^{(1)}, \phi_1^{(2)} \leq \frac{\pi}{2}$ ). Then the center intensities ( $\xi_1 + \xi_1^* + \Theta_1 = 0$ ) are  $|\Phi^{(1)}|_{center} = |\rho_1|\cos\phi_1^{(1)}, |\Phi^{(2)}|_{center} = |\rho_2|\cos\phi_1^{(2)}, -u = 2a_1^2$ . For the  $|\Phi^{(1)}|, |\Phi^{(2)}|$  components, in contrast to the background intensities  $\rho_1$  and  $\rho_2$ , their center intensities are much lower, which implies they are dark-dark solitons.

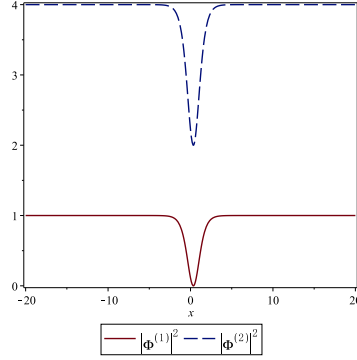
On the basis of the values of  $\alpha_1$  and  $\alpha_2$ , it can be divided into two different cases:

(1)  $\alpha_1 = \alpha_2$ , then  $K_1^{(1)} = K_1^{(2)}$ , therefore  $\phi_1^{(1)} = \phi_1^{(2)}$ , this indicates the  $\Phi^{(1)}$  and  $\Phi^{(2)}$  components are proportional to each other. Similar to the coupled NLS [18] and the coupled YO equations [3], the single dark-dark soliton for the coupled Maccari equations is identical to a scalar single dark soliton, which can be viewed as a degenerate case. We illustrate this situation in Fig.1. It is shown that at the soliton center both the  $\Phi^{(1)}$  and  $\Phi^{(2)}$  components are black.



**Figure 1.** The one dark-dark soliton solutions (degenerate) for the two-component Maccari system when  $t = 0$  with  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = \alpha_2 = 1, p_1 = 1 + i, \sigma_1 = -1, \sigma_2 = 1, \xi_{10R} = 0$ .

(2)  $\alpha_1 \neq \alpha_2$ , then  $K_1^{(1)} \neq K_1^{(2)}$ , therefore  $\phi_1^{(1)} \neq \phi_1^{(2)}$ , this suggests that the  $\Phi^{(1)}$  and  $\Phi^{(2)}$  components are not proportional mutually. So unlike the degenerate case, we can not simplify this case to a scalar dark soliton of the single component Maccari system. As is illustrated in Fig.2, at its center, the  $\Phi^{(1)}$  component is black, while the  $\Phi^{(2)}$  component is only gray.



**Figure 2.** The one dark-dark soliton solutions (non-degenerate) for the two-component Maccari system when  $t = 0$  with  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = 2, p_1 = 1 + i, \sigma_1 = -1, \sigma_2 = 1, \xi_{10R} = 0$ .

### 3.2. Two-soliton solution

To study the collision of two solitons, we take  $N = 2$  in the Eqs. (2.6)-(2.7). Then, we have

$$\Phi^{(1)} = \rho_1 e^{i(\alpha_1 x - \alpha_1^2 t)} \frac{g_2}{f_2}, \tag{3.7}$$

$$\Phi^{(2)} = \rho_2 e^{i(\alpha_2 x - \alpha_2^2 t)} \frac{h_2}{f_2}, \tag{3.8}$$

$$u = -2(\log f_2)_{xx}, \tag{3.9}$$

with

$$f_2 = 1 + e^{\xi_1 + \xi_1^* + \Theta_1} + e^{\xi_2 + \xi_2^* + \Theta_2} + \Omega_{12} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, \tag{3.10}$$

$$g_2 = 1 + K_1^{(1)} e^{\xi_1 + \xi_1^* + \Theta_1} + K_2^{(1)} e^{\xi_2 + \xi_2^* + \Theta_2} + \Omega_{12} K_1^{(1)} K_2^{(1)} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, \tag{3.11}$$

$$h_2 = 1 + K_1^{(2)} e^{\xi_1 + \xi_1^* + \Theta_1} + K_2^{(2)} e^{\xi_2 + \xi_2^* + \Theta_2} + \Omega_{12} K_1^{(2)} K_2^{(2)} e^{\xi_1 + \xi_1^* + \xi_2 + \xi_2^* + \Theta_1 + \Theta_2}, \tag{3.12}$$

and

$$e^{\Theta_j} = \frac{1}{p_j + p_j^*} = \frac{1}{2a_j}, \tag{3.13}$$

$$K_j^{(1)} = -\frac{p_j - i\alpha_1}{p_j^* + i\alpha_1} = -\frac{a_j + i(b_j - \alpha_1)}{a_j - i(b_j - \alpha_1)}, \quad (3.14)$$

$$K_j^{(2)} = -\frac{p_j - i\alpha_2}{p_j^* + i\alpha_2} = -\frac{a_j + i(b_j - \alpha_2)}{a_j - i(b_j - \alpha_2)}, \quad (3.15)$$

$$\Omega_{12} = \left| \frac{p_1 - p_2}{p_1 + p_2^*} \right|^2 = \frac{(a_1 - a_2)^2 + (b_1 - b_2)^2}{(a_1 + a_2)^2 + (b_1 - b_2)^2}, \quad (3.16)$$

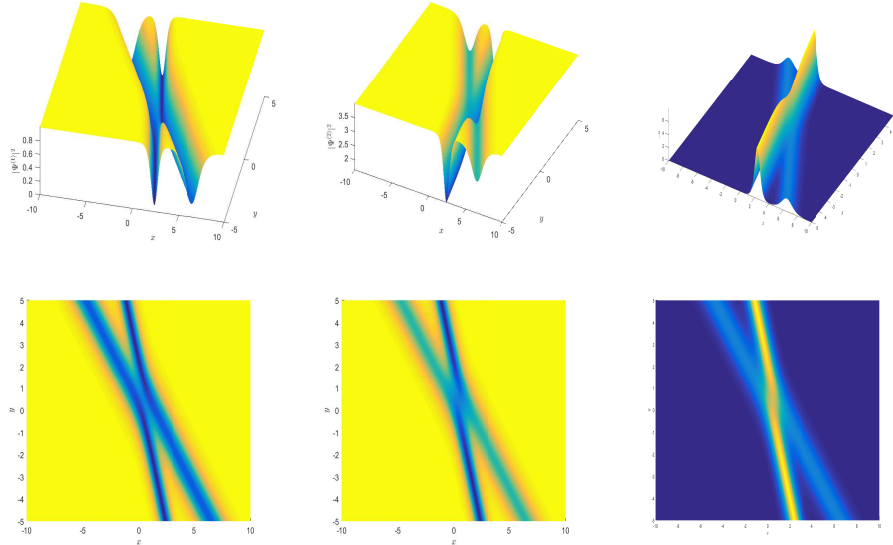
$$\begin{aligned} \xi_j + \xi_j^* &= 2a_j x + \left[ \frac{\sigma_1 \rho_1^2 a_j}{a_j^2 + (b_j - \alpha_1)^2} + \frac{\sigma_2 \rho_2^2 a_j}{a_j^2 + (b_j - \alpha_2)^2} \right] y - 4a_j b_j t + 2\xi_{j0R} \\ &= k_{x,j} x + k_{y,j} y + \omega_j t + 2\xi_{j0R}, \end{aligned} \quad (3.17)$$

where  $p_j = a_j + ib_j$ ,  $a_j, b_j, \xi_{j0R}$ , ( $j = 1, 2$ ) are real constants.

On this particular wave number of  $a_2 = -a_1$  and  $b_2 = b_1$ , i.e.,  $p_2 = -p_1^*$ , the denominator of  $\Omega_{12}$  is zero, that is to say,  $\Omega_{12}$  is singular. This choice will possess the  $Y$ -shape soliton interaction which is called the resonant solution in the KP system. In general, these two soliton interactions can be sorted into two different types: [4–6, 12]

(1) If  $a_1 a_2 < 0$ , then  $\Omega_{12} > 1$ . This soliton interaction is called  $O$ -type. The interaction peak (the maximum of  $-u$ ) is always greater than the sum of the asymptotic soliton amplitudes.

(2) If  $a_1 a_2 > 0$ , then  $0 < \Omega_{12} < 1$ . This soliton interaction is known as the  $P$ -type. The interaction peak (the maximum of  $-u$ ) is always less than the sum of the asymptotic soliton amplitudes.



**Figure 3.** The two dark-dark soliton solutions of the two-component Maccari system when  $t = 0$  with  $\rho_1 = 1$ ,  $\rho_2 = 2$ ,  $\alpha_1 = 1$ ,  $\alpha_2 = 2$ ,  $p_1 = 1 + \frac{1}{2}i$ ,  $p_2 = 2 + \frac{1}{3}i$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 1$ ,  $\xi_{10R} = \xi_{20R} = 0$ .



Obviously, the types of interaction do not rely on  $b_1$  and  $b_2$ . By taking the limit  $b_2 \rightarrow b_1$  into the two-soliton solution expressions of the equal-amplitude *O*-type interaction ( $a_1 = -a_2$ ), we can obtain the resonant *Y*-shape soliton solutions. In addition, resemble to Mel'nikov system, the interaction coefficient  $\Omega_{12}$  in the coupled Maccari system is always non-negative while for the KP equation,  $\Omega_{12}$  can be negative.

In Fig.3, we displayed the two dark-dark solitons collision. It is easy to observe that after collision, the two dark-dark solitons cross over each other without any change of darkness, velocity or shape in its two components. Hence, between the two solitons or between the  $\Phi^{(1)}$  and  $\Phi^{(2)}$  components after collision, there is no energy exchange. For the coupled Maccari equation, this kind of phenomenon is distinctly different from the bright-bright solitons collisions [15,30]. The bright two solitons interaction is elastic or inelastic, while there are only elastic interactions for the dark ones. This complete transmission of energy occurs for all combinations of  $\sigma_1, \sigma_2$  values.

## 4. Dark-dark soliton bound states

In this part, we explore the multi-dark-soliton bound states. In order to keep the constituent dark solitons staying together all the time, we take parameters

$$\frac{\omega_1}{k_{x,1}} = \frac{\omega_2}{k_{x,2}} \tag{4.1}$$

and

$$\frac{\omega_1}{k_{y,1}} = \frac{\omega_2}{k_{y,2}}. \tag{4.2}$$

### 4.1. The stationary dark-dark soliton bound states

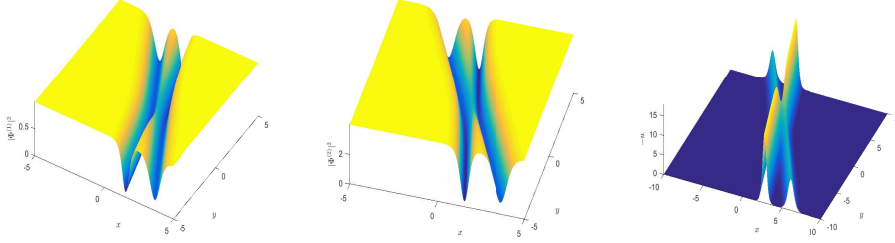
Through analysis, only when the common velocity equals to zero, the dark-dark solitons can form a stationary bound states, which requires

$$4a_i b_i = 0, (i = 1, 2). \tag{4.3}$$

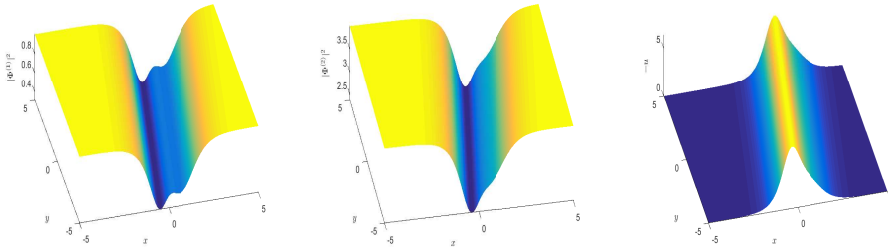
If  $a_j \leq 0$ , the soliton solutions would be singular, so we take  $a_j > 0$  to avoid this singularity, i.e.,  $b_i = 0$ . These constraints of the Maccari system can not be satisfied for all possible nonlinearity coefficients combinations, while for the CNLS [18] and the coupled YO system [3], only when  $\sigma_1$  and  $\sigma_2$  take opposite signs, the corresponding constraints are possible, and for the coupled Mel'nikov equation [10], the stationary solitons are possible for any combination of the nonlinearity coefficients.

In what follows, we illustrated two diverse examples of bound states. Corresponding to an oblique bound state, a nontrivial case of  $\frac{k_{y,1}}{k_{x,1}} \neq \frac{k_{y,2}}{k_{x,2}}$  is shown in Fig.4. Whereas, corresponding to a quasi-one-dimensional case, a trivial case of  $\frac{k_{y,1}}{k_{x,1}} = \frac{k_{y,2}}{k_{x,2}}$  is displayed in Fig.5.

For the coupled Maccari system, the nontrivial case of  $\frac{k_{y,1}}{k_{x,1}} \neq \frac{k_{y,1}}{k_{x,1}}$  of the stationary soliton bound states are possible for any combination of  $\sigma_1$  and  $\sigma_2$ . But in the trivial case of  $\frac{k_{y,1}}{k_{x,1}} = \frac{k_{y,1}}{k_{x,1}}$ , the bound states exist for only mixed types. The



**Figure 4.** A nontrivial case of the stationary dark-dark soliton bound states when  $\frac{k_{y,1}}{k_{x,1}} \neq \frac{k_{y,2}}{k_{x,2}}$  with the parameters  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = 1, p_1 = 2, p_2 = 3, \sigma_1 = 1, \sigma_2 = 1, \xi_{10R} = \xi_{20R} = 0$ .



**Figure 5.** A trivial case of the stationary dark-dark soliton bound states when  $\frac{k_{y,1}}{k_{x,1}} = \frac{k_{y,2}}{k_{x,2}}$  with the parameters  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = 2, p_1 = 1, p_2 = 2, \sigma_1 = 1, \sigma_2 = -1, \xi_{10R} = \xi_{20R} = 0$ .

reason is as follows. To obtain the trivial bound states, the parameters must satisfy the following condition:

$$\frac{\sigma_1 \rho_1^2}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{\sigma_2 \rho_2^2}{a_1^2 + (b_1 - \alpha_2)^2} = \frac{\sigma_1 \rho_1^2}{a_2^2 + (b_2 - \alpha_1)^2} + \frac{\sigma_2 \rho_2^2}{a_2^2 + (b_2 - \alpha_2)^2}, \quad (4.4)$$

$$b_1 = b_2 = 0.$$

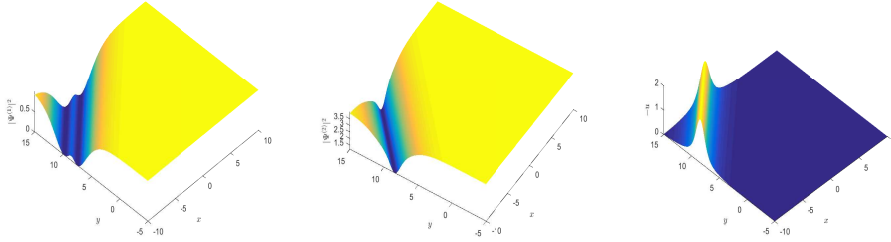
When  $\sigma_1$  and  $\sigma_2$  are both positive or negative, the left (or right) hand side of Eq.(4.4) is a decreasing or increasing function of  $a_j^2$ . So there is at most one  $a_j^2$  solution, hence at most one positive  $a_j$  value. This suggests that there are no stationary bound states when  $\sigma_1$  and  $\sigma_2$  are all focusing or defocusing. Nevertheless, when  $\sigma_1$  and  $\sigma_2$  are mixed, the left (or right) hand side may become non-monotone of  $a_j^2$ , hence it becomes possible for Eq.(4.4) to acquire two different positive values  $a_1$  and  $a_2$  when  $b_1 = b_2 = 0$ . In conclusion, the trivial-soliton bound states would occur only for mixed types.

## 4.2. The moving dark-dark soliton bound states

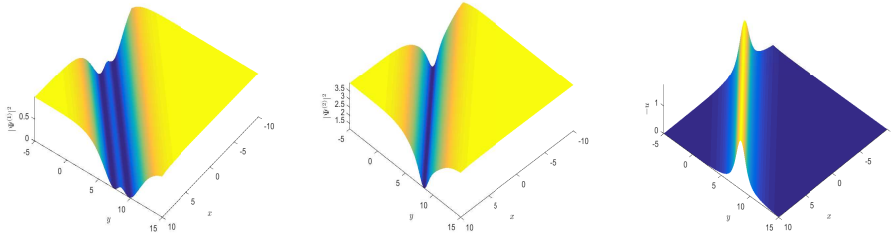
The common velocity of the moving bound states needs to be nonzero, i.e.,  $\omega_1 \neq 0$  and  $\omega_2 \neq 0$ . So we choose the parameters as

$$\frac{\sigma_1 \rho_1^2}{a_1^2 + (b_1 - \alpha_1)^2} + \frac{\sigma_2 \rho_2^2}{a_1^2 + (b_1 - \alpha_2)^2} = \frac{\sigma_1 \rho_1^2}{a_2^2 + (b_2 - \alpha_1)^2} + \frac{\sigma_2 \rho_2^2}{a_2^2 + (b_2 - \alpha_2)^2}, \quad b_1 = b_2. \quad (4.5)$$

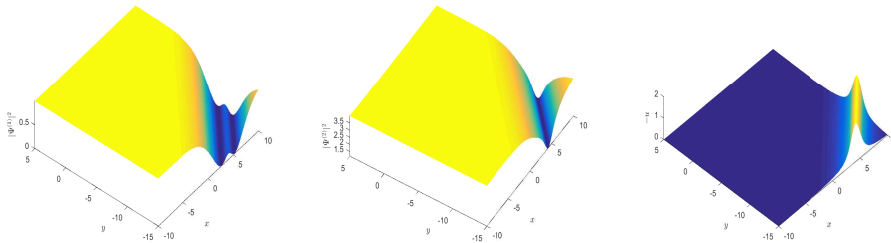
From the above expressions,  $\sigma_1$  and  $\sigma_2$  must be mixed types. The reason is the same as that shown in the stationary trivial bound states. It is obvious that when  $\sigma_1$  and  $\sigma_2$  take opposite signs, the left (right) hand side of Eq.(4.5) may be non-monotone of  $a_j^2$ , hence it becomes possible for Eq.(4.5) to acquire two different positive values  $a_1$  and  $a_2$  for the choice  $b_1 = b_2$ , which qualitatively resemble to the CNLS [18] and the coupled YO equation [3]. And the following figures 6-8 illustrate the profiles for the dark-dark soliton bound states of the moving case at different times.



**Figure 6.** The moving dark-dark soliton bound states when  $t = -10$  with the parameters  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = \frac{1}{2}, p_1 = 1 + i, p_2 = \frac{\sqrt{55}}{22} + i, \sigma_1 = 1, \sigma_2 = -1, \xi_{10R} = \xi_{20R} = 0$ .



**Figure 7.** The moving dark-dark soliton bound states when  $t = 0$  with the parameters  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = \frac{1}{2}, p_1 = 1 + i, p_2 = \frac{\sqrt{55}}{22} + i, \sigma_1 = 1, \sigma_2 = -1, \xi_{10R} = \xi_{20R} = 0$ .



**Figure 8.** The moving dark-dark soliton bound states when  $t = 10$  with the parameters  $\rho_1 = 1, \rho_2 = 2, \alpha_1 = 1, \alpha_2 = \frac{1}{2}, p_1 = 1 + i, p_2 = \frac{\sqrt{55}}{22} + i, \sigma_1 = 1, \sigma_2 = -1, \xi_{10R} = \xi_{20R} = 0$ .

It should be noted that both in the stationary and moving case of bound states, the phase shifts for the components  $\Phi^{(1)}$  and  $\Phi^{(2)}$  admit non-zero, while the com-

ponent  $u$  acquires no phase shift, when  $x$  and  $y$  vary from  $-\infty$  to  $+\infty$ . Actually, let  $2\phi_j^{(1)}$  and  $2\phi_j^{(2)}$  be the phases of constants  $K_j^{(1)}$  and  $K_j^{(2)}$ , thus  $\Phi_{phaseshift}^{(1)} = 2\phi_1^{(1)} + 2\phi_2^{(1)}$ ,  $\Phi_{phaseshift}^{(2)} = 2\phi_1^{(2)} + 2\phi_2^{(2)}$  and  $u_{phaseshift} = 0$  are the phase shifts of the components. We can find that the sum of the individual phase shifts of the two constituent dark solitons is equal to the total phase shifts of each short wave component, which is always non-zero, while in the  $u$  component they are generally zero. For example, as can be calculated from the above formula, the total phase shift of the  $\Phi^{(1)}$  is  $2\pi$ , and  $-3.4007$  in the  $\Phi^{(2)}$  component, they are all non-zero.

## 5. General $N$ dark soliton solutions of the multi-component Maccari system

In the same spirit as the coupled Maccari system, we can construct the  $N$  dark soliton solutions of the multi-component Maccari system. Therefore, the multi-component Maccari system consisting of  $M$  short wave components and one long wave component can be transformed to

$$(D_x^2 + 2i\alpha_k D_x + iD_t)g_k \cdot f = 0, k = 1, 2, \dots, M, \quad (5.1)$$

$$D_x D_y f \cdot f = \sum_{k=1}^M \sigma_k \rho_k^2 (f^2 - g_k g_k^*), \quad (5.2)$$

where  $f \equiv f(x, y, t)$  is a real,  $g_k \equiv g_k(x, y, t)$  are complex,  $\rho_k$  are constants,  $\alpha_k$  are real constants for  $k = 1, 2, \dots, M$ . Taking the determinants into account, we can explore the theorem as follows:

**Theorem 5.1.** *The  $N$ -dark soliton solutions of the multi-component Maccari system (5.1)-(5.2) are*

$$\Phi^{(k)} = \rho_k e^{i(\alpha_k x - \alpha_k^2 t)} \frac{g_k}{f}, u = -2(\log f)_{xx}, \quad (5.3)$$

where

$$f = \left| \delta_{ij} + \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N \times N},$$

$$g_k = \left| \delta_{ij} + \left( -\frac{p_i - i\alpha_k}{p_j^* + i\alpha_k} \right) \frac{1}{p_i + p_j^*} e^{\xi_i + \xi_j^*} \right|_{N \times N}, \quad (5.4)$$

with

$$\xi_j = p_j x + \frac{1}{2} \left( \sum_{k=1}^M \frac{\sigma_k \rho_k^2}{p_j - i\alpha_k} \right) y + ip_j^2 t + \xi_{j0},$$

where  $p_j$  and  $\xi_{j0}$  are complex constants.

The proof of Theorem 5.1 is omitted here.

## References

- [1] J.C. Chen, Y. Chen, B.F. Feng and K. Maruno, *General mixed multi-soliton solutions to one-dimensional multicomponent Yajima-Oikawa system*, Journal of the Physical Society of Japan, 2015, 84(7), 074001.
- [2] J.C. Chen, Y. Chen, B.F. Feng and K. Maruno, *Rational solutions to two- and one-dimensional multicomponent Yajima-Oikawa systems*, Physics Letters A, 2015, 379(24–25), 1510–1519.
- [3] J. Chen, Y. Chen, B.F. Feng and K. Maruno, *Multi-dark soliton solutions of the two-dimensional multi-component Yajima-Oikawa systems*, Journal of the Physical Society of Japan, 2015, 84(3), 034002.
- [4] S. Chakravarty and Y. Kodama, *Classification of the line-soliton solutions of KP II*, Journal of Physics A: Mathematical and Theoretical, 2008, 41(27), 275209.
- [5] S. Chakravarty and Y. Kodama, *Soliton solutions of the KP equation and application to shallow water waves*, Studies in Applied Mathematics, 2009, 123(1), 83–151.
- [6] S. Chakravarty, T. Lewkow and K.I. Maruno, *On the construction of the KP line-solitons and their interactions*, Applicable Analysis, 2010, 89(4), 529–545.
- [7] B.F. Feng, *General N-soliton solution to a vector nonlinear Schrödinger equation*, Journal of Physics A: Mathematical and Theoretical, 2014, 47(35), 355203.
- [8] R. Hirota, *The direct method in soliton theory*, Cambridge University Press, Cambridge, 2004.
- [9] Z. Han and Y. Chen, *Bright-dark mixed N-soliton solution of the two-dimensional Maccari system*, Chinese Physics Letters, 2017, 34(7), 070202.
- [10] Z. Han, Y. Chen and J. Chen, *General N-dark soliton solutions of the multi-component Mel'nikov system*, Journal of the Physical Society of Japan, 2017, 86(7), 074005.
- [11] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publications of the Research Institute for Mathematical Sciences, 1983, 19(3), 943–1001.
- [12] Y. Kodama, *KP solitons in shallow water*, Journal of Physics A: Mathematical and Theoretical, 2010, 43(43), 434004.
- [13] D.W.C. Lai and K.W. Chow, *Coalescence of ripples, breathers, dromions and dark solitons*, Journal of the Physical Society of Japan, 2001, 70(3), 666–677.
- [14] M. Z. Li, B. Tian and Q.X. Qu, et al, *Dark-dark solitons for a coupled variable-coefficient higher-order nonlinear Schrödinger system in an inhomogeneous optical fiber*. Superlattices and Microstructures, 2017, 112, 20–29.
- [15] L. Liu, B. Tian, Y.Q. Yuan and Y. Sun, *Bright and dark N-soliton solutions for the (2+1)-dimensional Maccari system*, The European Physical Journal Plus, 2018, 133(2), 72.
- [16] A. Maccari, *Universal and integrable nonlinear evolution systems of equations in 2+1 dimensions*, Journal of Mathematical Physics, 1997, 38(8), 4151–4164.
- [17] Y. Ohta, *Wronskian solutions of soliton equations*, RIMS Kokyuroku, 1989, 684.

- [18] Y. Ohta, D.S. Wang and J. Yang, *General  $N$ -dark-dark solitons in the coupled nonlinear Schrödinger equations*, Studies in Applied Mathematics, 2011, 127(4), 345–371.
- [19] Y.B. Shi and Y. Zhang, *Rogue waves of a  $(3+1)$ -dimensional nonlinear evolution equation*, Communications in Nonlinear Science and Numerical Simulation, 2017, 44, 120–129.
- [20] A. Uthayakumar, K. Nakkeeran and K. Porsezian, *Soliton solution of new  $(2+1)$  dimensional nonlinear partial differential equations*, Chaos, Solitons and Fractals, 1999, 10(9), 1513–1518.
- [21] R. Willox and I. Loris, *An algebraic description of generalized  $k$ -constraints*, Journal of Physics A: Mathematical and General, 1999, 32(10), 2027.
- [22] R. Willox and I. Loris, *KP constraints from reduced multi-component hierarchies*, Journal of Mathematical Physics, 1999, 40(12), 6501–6525.
- [23] G.H. Wang, L.H. Wang, J.G. Rao and J.S. He, *New patterns of the two-dimensional rogue waves:  $(2+1)$ -dimensional Maccari system*, Communications in Theoretical Physics, 2017, 67(6), 601.
- [24] F. Yuan, J.G. Rao, K. Porsezian, D. Mihalache and J.S. He, *Various exact rational solutions of the two-dimensional Maccari's system*, Romanian Journal of Physics, 2016, 61, 378–399.
- [25] Y.Q. Yuan, B. Tian and L. Liu, et al, *Dark-dark solitons for a set of the coupled nonlinear Schrödinger equations in a birefringent fiber*, Chaos, Solitons and Fractals, 2018, 107, 216–221.
- [26] G.F. Yu and Z.W. Xu, *Dynamics of a differential-difference integrable  $(2+1)$ -dimensional system*, Physical Review E, 2015, 91(6), 062902.
- [27] J.F. Zhang, *Generalized dromion structures of new  $(2+1)$ -dimensional nonlinear evolution equation*, Communications in Theoretical Physics, 2001, 35(3), 267.
- [28] J.F. Zhang, W.H. Huang and C.L. Zheng, *Coherent soliton structures of a new  $(2+1)$ -dimensional evolution equation*, Acta Physic Sinica, 2002, 51, 2676–2682 (in Chinese).
- [29] Y. Zhang, Y.B. Sun and W. Xiang, *The rogue waves of the KP equation with self-consistent sources*, Applied Mathematics and Computation, 2015, 263, 204–213.
- [30] H.L. Zhen, B. Tian and W.R. Sun, *Dynamics of an integrable Kadomtsev-Petviashvili-based system*, Applied Mathematics Letters, 2014, 27, 90–96.