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Global Phase Portraits of Symmetrical Cubic Hamiltonian Systems with a Nilpotent Singular Point*

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Abstract Han et al. [Han et al., *Polynomial Hamiltonian systems with a nilpotent critical point*, J. Adv. Space Res. 2010, 46, 521–525] successfully studied local behavior of an isolated nilpotent critical point for polynomial Hamiltonian systems. In this paper, we extend the previous result by analyzing the global phase portraits of polynomial Hamiltonian systems. We provide 12 non-topological equivalent classes of global phase portraits in the Poincaré disk of cubic polynomial Hamiltonian systems with a nilpotent center or saddle at origin under some conditions of symmetry.

Keywords Hamiltonian systems, nilpotent singular point, global phase portraits, Poincaré transformation.

MSC(2010) 70H05, 34C05.

1. Introduction

Hamiltonian systems are relevant to a variety of space and astrophysical studies such as celestial mechanics, cosmology and nonlinear plasma waves. In this paper, we study polynomial Hamiltonian systems with a nilpotent critical point. Let $H(x,y)$ be a real polynomial in (x,y). Then, as we know, a system of the form

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.1}$$

is called a polynomial Hamiltonian system. There have been many studies on the number of limit cycles of various perturbed systems to the form of system (1.1) by using the method of Melnikov functions. For the unperturbed system (1.1), one usually supposes that it has a period annulus consisting of a family of periodic orbits with its boundary containing an elementary center point or a hyperbolic saddle point. Han et al. [9] give a complete classification to nilpotent critical points for the polynomial Hamiltonian system (1.1) with exact three different types of

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a center, a cusp or a saddle. Then for quadratic and cubic Hamiltonian systems they obtain necessary and sufficient conditions for a nilpotent critical point to be a center, a cusp or a saddle. They also give local phase portraits of this kind of these systems under some conditions of symmetry.

Recently Colak, Llibre and Valls [1–4] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. The complete classification of the phase portrait of the nilpotent centers in the last case was given in [5]. The quasi-homogeneous but non-homogeneous polynomial differential systems have also been investigated from different aspects, for example, the structural stability, the integrability, the polynomial and rational integrability, the centers and limit cycles, the normal forms. Recently, García et al. [8] provide an algorithm for obtaining all the quasi-homogeneous but not homogeneous polynomial differential systems with a given degree. Using this algorithm they obtain all the quadratic and cubic quasi-homogeneous but not homogeneous vector fields. Liang et al. [10] give a complete classification of the global phase portraits of planar quasi-homogeneous but not homogeneous polynomial differential systems of degree 4. More recently, Dias, Llibre, Valls [6] classify the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three symmetric with respect to the x-axis having a nilpotent center at the origin.

In this paper, we extend the previous result in [9] by analyzing the global phase portraits of polynomial Hamiltonian systems. We provide 12 non-topological equivalent classes of global phase portraits in the Poincaré disk of cubic polynomial Hamiltonian systems with a nilpotent center or saddle at origin under some conditions of symmetry.

2. Main results

For the cubic polynomial Hamiltonian system

$$H(x,y) = \frac{y^2}{2} + \sum_{3\le i+j\le 4} h_{ij}x^i y^j, \tag{2.1}$$

where i, j are natural number, Han et al. give a complete classification for the nilpotent singular point (see Theorem 2 in [9]). In the following suppose (2.1) holds with

$$H(\pm x, \pm y) = H(x,y).$$

By (1.1) and (2.1) we can obtain

$$H(x,y) = \frac{y^2}{2} + cx^4 + ax^2y^2 + by^4,$$

with $a = h_{22}$, b=h_{04}, c=h_{40}. We can get the system

$$\begin{cases} \dot{x} = y(1 + 2ax^2 + 4by^2), \\ \dot{y} = -2x(ay^2 + 2cx^2). \end{cases} \tag{2.2}$$

Han et al. [9] show that the origin is a nilpotent saddle if $c < 0$, and when $c > 0$, the origin is a nilpotent center. Our main result is the following

Theorem 2.1. *The global phase portraits of system (2.2) with a nilpotent saddle or nilpotent center at the origin are topologically equivalent to one of the phase portraits given in Fig. 1.*

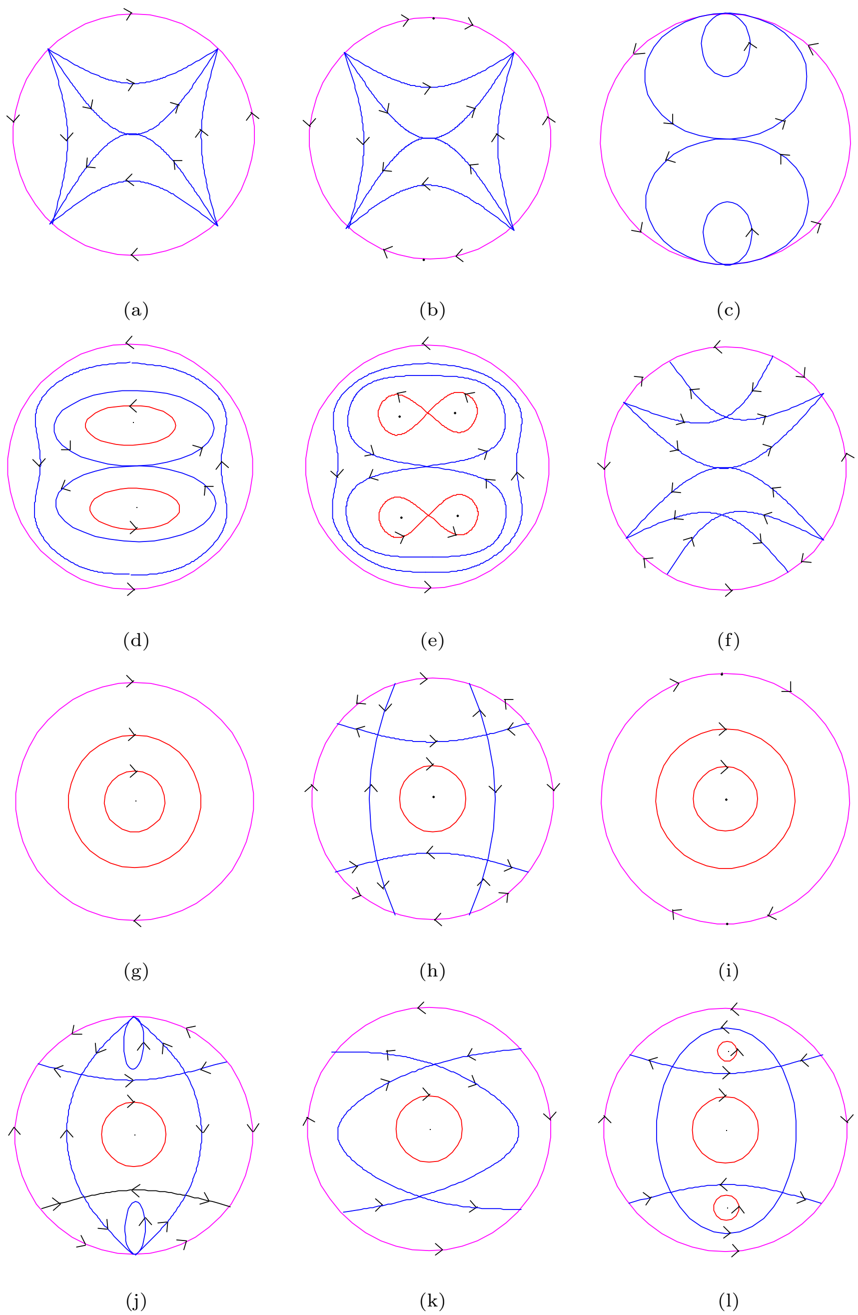


Figure 1. Global phase portraits of system (2.2).

3. The Poincaré compactification

In order to study the distribution of trajectories of plane autonomous systems on the whole plane, besides the singular point of the system in the finite plane, the orbit of the system is also studied at infinity. We introduce Poincaré compactification. For more detail on the Poincaré compactification please refer to Chapter 5 of [7]. Let S^2 be the set of points $(s_1, s_2, s_3) \in R^3$ such that $s_1^2 + s_2^2 + s_3^2 = 1$, called the Poincaré sphere. Given a polynomial vector field

$$X(x, y) = (\dot{x}, \dot{y}) = (P(x, y), Q(x, y))$$

in R^2 of degree d (where d is the maximum of the degrees of the polynomials P and Q) it can be extended analytically to the Poincaré sphere by projecting each point $x \in R^2$ identified with the point $(x_1, x_2, 1) \in R^3$ in the Poincaré sphere using a straight line through x and the origin of R^3. The equator $S^1 = \{(s_1, s_2, s_3) \in S^2 : s_3 = 0\}$ corresponds to the infinity of R^2. In this way we obtain a vector field $\hat{X}$ in $S^2 \setminus S^1$. The vector field $\hat{X}$ is formed by two copies of X: one on the northern hemisphere $\{(s_1, s_2, s_3) \in S^2 : s_3 > 0\}$ and another on the southern hemisphere $\{(s_1, s_2, s_3) \in S^2 : s_3 < 0\}$. The local charts need for doing the calculations on the Poincaré sphere are

$$U_i = \{s \in S^2 : s_i > 0\}, \qquad V_i = \{s \in S^2 : s_i < 0\}$$

where $s = (s_1, s_2, s_3)$, with the corresponding local maps

$$\varphi_i(s) : U_i \to R^2, \qquad \psi_i(s) : V_i \to R^2,$$

such that $\varphi_i(s) = -\psi_i(s) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$ $(i = 1, 2, 3)$. The expression for the corresponding vector field on S^2 in the local chart U_1 is given by

$$\begin{cases} \dot{u} = v^d[-uP(1/v, u/v) + Q(1/v, u/v)], \\ \dot{z} = -v^{d+1}P(1/v, u/v); \end{cases} \tag{3.1}$$

the expression for U_2 is

$$\begin{cases} \dot{u} = v^d[-uQ(u/v, 1/v) + P(u/v, 1/v)], \\ \dot{v} = -v^{d+1}Q(u/v, 1/v); \end{cases} \tag{3.2}$$

and the expression for U_3 is

$$\begin{cases} \dot{u} = P(u, v), \\ \dot{v} = -Q(u, v). \end{cases}$$

The expressions for the charts V_i are those for the charts U_i multiplied by $(-1)^{d-1}$, for $i = 1, 2, 3$. Hence for studying the vector field X it is enough to study its Poincaré compactification restricted to the northern hemisphere plus S^1. To draw the phase portraits we consider the projection by $\pi(s_1, s_2, s_3) = (s_1, s_2)$ of the closed northern hemisphere in to the local disk $D = \{(s_1, s_2) : s_1^2 + s_2^2 \leq 1\}$, called the Poincaré disk.

Finite singular points of X are the singular points of its compactification which are in S^2/S^1, and they can be studied using U_3. Infinite singular points of X, on the other hand, are the singular points of the corresponding vector field on the Poincaré disk lying on S^1. Note that if $s \in S^1$ is an infinite singular point, then $-s$ is also an infinite singular point. Hence to study the infinite singular points it suffices to look for them only at $U_{1|v=0}$ and at the origin of U_2.

4. Finite singular points

From Han et al. [9], we can know the types of finite singular points of system (2.2).

When $x = 0$, $y \neq 0$ and $b < 0$, we see that the finite singular points of system (2.2) other than its origin are

$$A_{1,2} = (0, \pm 1/(2\sqrt{-b})).$$

When $x \neq 0$, $y \neq 0$ and $ac < 0$, we see that the finite singular points of system (2.2) other than its origin are

$$A_{3,4} = (\sqrt{\frac{a}{2(4bc - a^2)}}, \pm\sqrt{\frac{c}{a^2 - 4bc}}),$$

$$A_{5,6} = (-\sqrt{\frac{a}{2(4bc - a^2)}}, \pm\sqrt{\frac{c}{a^2 - 4bc}}).$$

We classify a singular point q, whether it is a finite singular point or an infinite singular point, as follows:

q is hyperbolic if its two eigenvalues have non-zero real part. The local phase portraits of hyperbolic singular points can be obtained from Theorem 2.15 of [7];

q is semi-hyperbolic if one of its eigenvalues is zero and the other is non-zero. The local phase portraits of semi-hyperbolic singular points are characterized in Theorem 2.19 of [7];

q is nilpotent if its two eigenvalues are zero but its linear part is not identically zero. The local phase portraits of nilpotent singular points can be obtained from Theorem 3.15 of [7];

q is degenerate if its linear part is identically zero. The local phase portraits of such singular points are studied doing changes of variables called blow-ups, see Chapters 2 and 3 of [7].

5. Global phase portraits of system (2.2)

From Han et al. [9], we can find the types of finite singular points of system (2.2). In this section, we will only analyze infinite singular points and draw global phase portraits of system (2.2).

(i)When $c < 0$, the origin is a nilpotent saddle.

For system (2.2), the following six cases to discuss:

(ia)$b > 0$, $a > 0$

Only $O(0,0)$ is a finite singular point, and $O(0,0)$ is a nilpotent saddle.

We will discuss the infinite singular points of the system (2.2) in the following. Using (3.1), we see that in the local chart U_1 system (2.2) become

$$\begin{cases} \dot{u} = -4au^2 - u^2v^2 - 4bu^4 - 4c, \\ \dot{v} = -uv^3 - 2auv - 4bu^3v. \end{cases} \tag{5.1}$$

When $v = 0$, there are two solutions on U_1: $B_1(\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$, $B_2(-\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$. The linear part of system (5.1) at the points $(u, 0)$ is

$$\begin{pmatrix} -8u(a+2bu^2) & 0 \\ 0 & -2u(a+2bu^2) \end{pmatrix}.$$

The both of the eigenvalues of two singular points are negative if $u > 0$, and positive if $u < 0$. Hence the singular points B_1 and B_2 are stable and unstable nodes, respectively.

Using (3.2), we see that in the local chart U_2 system (2.2) become

$$\begin{cases} \dot{u} = v^2 + 4au^2 + 4b + 4cu^4, \\ \dot{v} = 2auv + 4cu^3v. \end{cases} \tag{5.2}$$

When $b \neq 0$, the origin is not a singular point of the system (5.2). We say that two vector fieldsdisk are topologically equivalent on the Poincaré if there exists a homeomorphism from one onto the other which sends orbits to orbits preserving or reversing the direction of the flow. When $a \leq 0$, the global phase portrait of system (2.2) is topologically equivalent to the phase portrait (a) of Fig. 1.

The global phase portrait is shown in (a) of Fig. 1.

(ib)$b = 0$, $a > 0$

Only $O(0,0)$ is a finite singular point, and $O(0,0)$ is a nilpotent saddle. The following is a discussion of the infinite singular point of the system. Using (3.1), we see that in the local chart U_1 system (2.2) becomes

$$\begin{cases} \dot{u} = -4au^2 - u^2v^2 - 4c, \\ \dot{v} = -uv^3 - 2auv. \end{cases} \tag{5.3}$$

When $v = 0$, the singular points of system (5.3) is $B_1(\sqrt{-c/a}, 0)$ and $B_2(-\sqrt{-c/a}, 0)$. The linear part of system (5.3) is

$$\begin{pmatrix} -8ua & 0 \\ 0 & -2au \end{pmatrix}.$$

Hence the singular points $B_1(\sqrt{-c/a}, 0)$ and $B_2(-\sqrt{-c/a}, 0)$ are stable and unstable nodes, respectively. Using (3.2), we see that in the local chart U_2 system (2.2) becomes

$$\begin{cases} \dot{u} = v^2 + 4au^2 + 4cu^4, \\ \dot{v} = 2auv + 4cu^3v. \end{cases} \tag{5.4}$$

When $b = 0$, the origin is a singular point of the system (5.4). We see that the origin is a degenerate singular point. We need to do blow-up to understand the local behavior at the point. We perform the directional blow-up $(u, v) \to (u, w)$ with

$$u = u, \qquad w = \frac{v}{u}.$$

and have

$$\begin{cases} \dot{u} = w^2u^2 + 4au^2 + 4cu^4, \\ \dot{w} = -2auw - uw^3. \end{cases} \tag{5.5}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and get the vector field

$$\begin{cases} \dot{u} = w^2u + 4au^2 + 4cu^3, \\ \dot{w} = -2aw - w^3. \end{cases} \tag{5.6}$$

When $u = 0$, system (5.6) have the unique singular point $(0, 0)$ when $a > 0$. We note that the linear part of system (5.6) at the origin is

$$\begin{pmatrix} \text{4a} & 0 \\ 0 & \text{-2a} \end{pmatrix}.$$

The eigenvalues of the linear part of system (5.6) at the origin are $4a$ and $-2a$, hence the origin is a saddle. Going back through the change of variables until system (5.4) as shown in system (5.4) of Fig. 2. We see that locally the origin of U_2 consists of hyperbolic sectors.

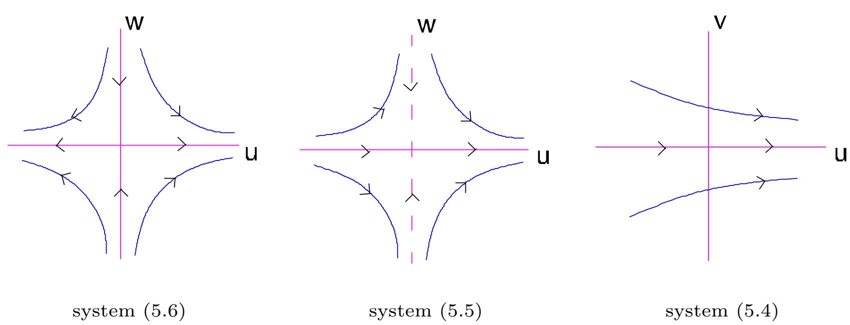


Figure 2. Blow-up of the origin of U_2 of system (5.4).

The global phase portrait is shown in (b) of Fig. 1.

(ic)$b = 0$, $a \leq 0$

When $b = 0$ and $a = 0$ the finite singular point is only $O(0, 0)$, and $O(0, 0)$ is a nilpotent saddle.

The following is a discussion of the infinite singular point of the system (2.2). Using (3.1), we see that in the local chart U_1 system (2.2) becomes

$$\begin{cases} \dot{u} = -u^2v^2 - 4c, \\ \dot{v} = -uv^3. \end{cases} \tag{5.7}$$

When $v = 0$, there are no singular points on the local chart U_1.

Using (3.2), we see that in the local chart U_2 system (2.2) becomes

$$\begin{cases} \dot{u} = v^2 + 4cu^4, \\ \dot{v} = 4cu^3 v. \end{cases} \tag{5.8}$$

When $b = 0$, the origin is a singular point of the system (5.8). We see that the origin is a degenerate singular point. We need to do blow-up to understand the local behavior at the point. We perform the directional blow-up $(v, z) \to (v, w)$ with

$$u = u, \quad w = \frac{v}{u},$$

and have

$$\begin{cases} \dot{u} = w^2 u^2 + 4cu^4, \\ \dot{w} = -uw^3. \end{cases}$$

We eliminate the common factor u between $\dot{u}$ and $\dot{w}$, and get the vector field

$$\begin{cases} \dot{u} = uw^2 + 4cu^3, \\ \dot{w} = -w^3. \end{cases} \tag{5.9}$$

We see that the origin is still a degenerate singular point. We need to do blow-up to understand the local behavior at the point. We perform the directional blow-up $(u, w) \to (u, z)$ with

$$u = u, \quad z = \frac{w}{u},$$

and have

$$\begin{cases} \dot{u} = z^2 u^3 + 4cu^3, \\ \dot{z} = -2z^3 u^2 - 4cu^2 z. \end{cases}$$

We eliminate the common factor u^2 between $\dot{u}$ and $\dot{w}$, and get the vector field

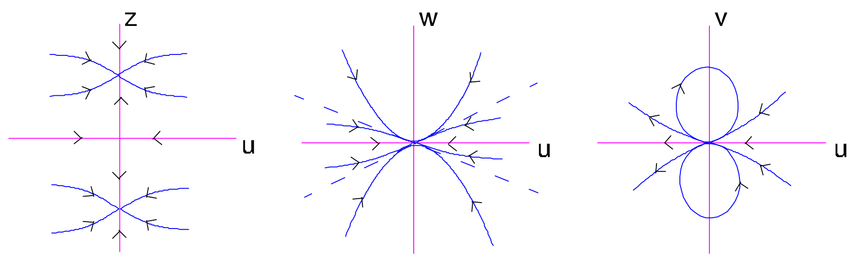
$$\begin{cases} \dot{u} = z^2 u + 4cu, \\ \dot{z} = -2z^3 - 4cz. \end{cases} \tag{5.10}$$

When $u = 0$, the possible singular points of system (5.10) are $(0, 0)$, $(0, \sqrt{-2c})$ and $(0, -\sqrt{-2c})$. The linear part of system (5.10) at the points $(0, \pm\sqrt{-2c})$ is

$$\begin{pmatrix} 2c & 0 \\ 0 & 8c \end{pmatrix}.$$

The both of the eigenvalues of two singular points are negative. Hence the singular points B_1 and B_2 are stable nodes.

The eigenvalues of the linear part of system (5.10) at the point $(0, 0)$ are $\pm 4c$, hence it is a saddle. Then, tracing back the change of variables to system (5.8), see Fig. 3. We see that locally the origin of U_2 has two elliptic sectors and two parabolic sectors.



system (5.10) system (5.9) system (5.8)

Figure 3. Blow-up of the origin of U_2 of system (5.8).

The global phase portrait is shown in (c) of Fig. 1.

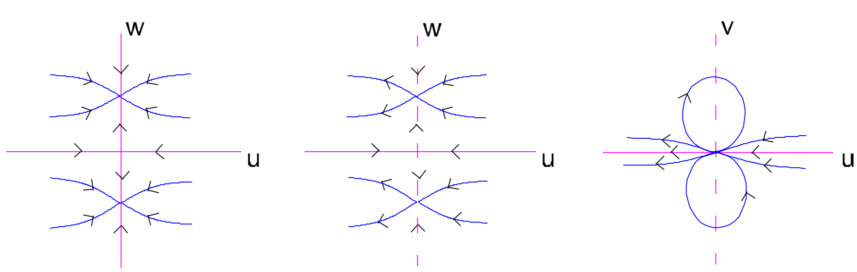
When $b = 0$, $a < 0$ the finite singular point is only $O(0,0)$, and $O(0,0)$ is a nilpotent saddle.

The following is a discussion of the infinite singular point of the system (2.2). Using (3.1), we see that in the local chart U_1 system (2.2) becomes system (5.3). When $v = 0$, system (5.3) has no singular point since $a < 0$.

Using (3.2), we see that in the local chart U_2 system (2.2) becomes system (5.4). When $b = 0$, the origin is a singular point of the system (5.4). We see that the origin is a degenerate singular point. We need to do blow-up to understand the local behavior at the point.We can get system (5.6). When $u = 0$ the singular points of system (5.6) are $(0,0)$,$(0, \pm\sqrt{-2a})$. The linear part of system (2.2) at the points $(0, w)$ is

$$\begin{pmatrix} w^2 + 4a & 0 \\ 0 & -2a - 3w^2 \end{pmatrix}.$$

So the points $(0, \pm\sqrt{-2a})$ are both stable nodes. The origin is a saddle. Then, tracing back the change of variables to system (5.4), see Fig. 4. Therefore, the origin of U_2 has two elliptic sectors and two parabolic sectors in this case.



system (5.6) system (5.5) system (5.4)

Figure 4. Blow-up of the origin of U_2 of system (5.4).

When $b = 0$ and $a < 0$ the global phase portrait of system (2.2) are topologically equivalent to the phase portrait (c) of Fig. 1.

(id)$b < 0$, $a \leq 0$

When $a < 0$, $O(0,0)$ is a nilpotent saddle. $A_1 = (0, 1/2\sqrt{-b})$ and $A_2 = (0, -1/2\sqrt{-b})$ are all centers.

When $a = 0$, we see that A_1 and A_2 are degenerate singular points. We need to do blow-up to understand the local behavior at the two points. First we translate A_1 to the origin. The system (2.2) becomes

$$\begin{cases} \dot{x} = y - 2by^3 - 3\sqrt{-b}y^2, \\ \dot{y} = 2cx^3. \end{cases} \tag{5.11}$$

The system has the Hamiltonian

$$H(x,y) = \frac{1}{2}y^2 - \frac{1}{2}by^4 - \sqrt{-b}y^3 - \frac{c}{2}x^4.$$

In [9], h_{30} and h_{40} represent the coefficient of x^3y^0 and x^4y^0, respectively. We can know that A_1 is a center when $h_{30} = 0$, $h_{40} = -c/2 > 0$.

We translate A_2 to the origin. The system (2.2) becomes system (5.8), so A_2 is also a center. We see that in the local chart U_1 plane, $bu^4 + au^2 + c = 0$ has no solution, and in the U_2 plane the origin is not a solution, so there is no infinite singular point.

The global phase portrait is shown in (d) of Fig. 1.

(ie)$b < 0$, $0 < a < 2\sqrt{bc}$

$O(0,0)$ is a nilpotent saddle. A_1 and A_2 are all saddles. $A_i (i = 3,4,5,6)$ are all centers. If $b < 0$ and $0 < a < 2\sqrt{bc}$, we see that the equation $bu^4 + au^2 + c = 0$ has no solution in the local chart U_1 plane, and $D(0,0)$ is not a solution in the U_2 plane. So there is no infinite singular point.

The global phase portrait is shown in (e) of Fig. 1.

(if)$b < 0$, $a \geq 2\sqrt{bc}$

$O(0,0)$ is a nilpotent saddle. A_1 and A_2 are all saddles.

When $v = 0$, there are four solutions $B_{1,2}(\sqrt{\pm\sqrt{(a^2-4bc)/4b^2} - a/2b}, 0)$, $B_{3,4}(-\sqrt{\pm\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$ on U_1. The linear part of system (5.1) at the point $(u,0)$ is

$$\begin{pmatrix} -8u(a+2bu^2) & 0 \\ 0 & -2u(a+2bu^2) \end{pmatrix}.$$

The eigenvalues of the linear part of system (5.1) are $-8u(a+2bu^2)$ and $-2u(a+2bu^2)$ at the point $(u,0)$. Hence the singular points B_1 and B_4 are unstable nodes and B_2 and B_3 are stable nodes. We see that the origin is not a solution in the U_2 plane.

The global phase portrait is shown in (f) of Fig. 1.

(ii)When $c > 0$, the origin is a nilpotent center. For system (2.2), the following six cases to discuss:

(iia)$b > 0$, $a \geq -2\sqrt{bc}$

The finite singular point is only $O(0,0)$, and $O(0,0)$ is a nilpotent center.If $b > 0$ and $a \geq -2\sqrt{bc}$ we see that in the local chart U_1 plane, $bu^4 + au^2 + c = 0$ has no solution, and in the U_2 plane the origin is not a solution, so there is no infinite singular point.

The global phase portrait is shown in (g) of Fig. 1.

(iib)$b > 0$, $a < -2\sqrt{bc}$

The finite singular points are $O(0,0)$, $A_i(i=3,4,5,6)$. $O(0,0)$ is a nilpotent center. $A_i(i=3,4,5,6)$ are all saddles.

We see that in U_1 plane, there has solutions $B_{1,2}(\sqrt{\pm\sqrt{(a^2-4bc)/4b^2}-a/2b},0)$, $B_{3,4}(-\sqrt{\pm\sqrt{(a^2-4bc)/4b^2}-a/2b},0)$. The linear part of system (5.1) at the point $(u,0)$ is

$$\begin{pmatrix} -8u(a+2bu^2) & 0 \\ 0 & -2u(a+2bu^2) \end{pmatrix}.$$

The eigenvalues of the linear part of system (5.1) at the points $(u,0)$ are $-8u(a+2bu^2)$ and $-2u(a+2bu^2)$. Therefore, the singular points B_i $(i=1,4)$ and B_i $(i=2,3)$ are stable and unstable nodes, respectively. We see that in the U_2 plane the origin is not a solution.

The global phase portraits is shown in (h) of Fig. 1.

(iic)$b=0$, $a\geq 0$

The finite singular point is only $O(0,0)$, and $O(0,0)$ is a nilpotent center. We see that in the local chart U_1 plane, $bu^4+au^2+c=0$ has no solution, and in the U_2 plane the origin is a solution. The eigenvalues of the linear part of system (5.6) at the origin are $4a$ and $-2a$, hence the origin is a saddle. Going back through the change of variables until system (5.4) as shown in system (5.4) of Fig. 2. We see that locally the the origin of U_2 consists of hyperbolic sectors.

The global phase portrait is shown in (i) of Fig. 1.

(iid)$b=0$, $a<0$

The finite singular points are $O(0,0)$, $A_i(i=3,4,5,6)$. $O(0,0)$ is a nilpotent center. $A_i(i=3,4,5,6)$ are all saddles. We see that in the local chart U_1 plane, there have two solutions $B_1(\sqrt{-c/a},0)$ and $B_2(-\sqrt{-c/a},0)$. When $v=0$ and $a<0$, the singular points of system (5.3) are $B_1(\sqrt{-c/a},0)$ and $B_2(-\sqrt{-c/a},0)$. The linear part of system (5.3) is

$$\begin{pmatrix} -8ua & 0 \\ 0 & -2au \end{pmatrix}.$$

When $a<0$, the eigenvalues of the linear part of system (5.3) at the points $(u,0)$ are $-8au$ and $-2au$. So, the singular points B_1 and B_2 are stable and unstable nodes, respectively.

We see that the origin is a solution and the origin is a degenerate singular point in the U_2 plane. We need to do blow-up to understand the local behavior at the point. Similar to (ib) situation we can get system (5.6).

When $u=0$, the possible singular points of system (5.6) are $(0,0)$, $(0,\sqrt{-2a})$ and $(0,-\sqrt{-2a})$.

When $a<0$, the singular points of system (5.6) are $(0,\pm\sqrt{-2a})$,and they are all stable nodes.

The point $(0,0)$ is a saddle because the eigenvalues of system (5.7) at the point are $4a$ and $-2a$. Then, tracing back the change of variables to system (5.4), see Fig. 4.

The global phase portraits is shown in (j) of Fig. 1.

(iie)$b<0$, $a\geq 0$

The finite singular points are O(0, 0), A_1 and A_2.

O(0, 0) is a nilpotent saddle. $A_1 = (0, 1/2\sqrt{-b})$ and $A_2 = (0, -1/2\sqrt{-b})$ are all saddles.

In U_1 plane, there have two solutions $B_1(\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$, $B_2(-\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$. The linear part of system (5.1) at the points $(u, 0)$ is

$$\begin{pmatrix} -8u(a+2bu^2) & 0 \\ 0 & -2u(a+2bu^2) \end{pmatrix}.$$

The eigenvalues of the linear part of system (5.1) at the points $(u, 0)$ are $8u\sqrt{a^2-4bc}$ and $2u\sqrt{a^2-4bc}$. Therefore, the singular points B_1 and B_2 are unstable and stable nodes, respectively.

When $a = 0$, we see that A_1 and A_2 are degenerate singular points. We need to do blow-up to understand the local behavior at the two points. Firstly, we translate A_1 to the origin. The system (2.2) becomes

$$\begin{cases} \dot{x} = y - 2by^3 - 3\sqrt{-b}y^2, \\ \dot{y} = 2cx^3. \end{cases}$$

The system has the Hamiltonian

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{2}by^4 - \sqrt{-b}y^3 - \frac{c}{2}x^4.$$

In [9], we can know that A_1 is a saddle when $h_{30} = 0$, $h_{40} = -c/2 < 0$. First we translate A_1 to the origin. A_1 and A_2 are all saddles.

We see that in the local chart U_1 plane, there are two solutions $B_1(\sqrt{\sqrt{-c/b}}, 0)$ and $B_2(-\sqrt{\sqrt{-c/b}}, 0)$. The linear part of system (5.1) at the point $(u, 0)$ is

$$\begin{pmatrix} -16bu^3 & 0 \\ 0 & -4bu^3 \end{pmatrix}.$$

Therefore, the singular points B_1 and B_2 are unstable and stable nodes, respectively.

We see that in the U_2 plane the origin is not a solution.

The global phase portrait is shown in (k) of Fig. 1.

(iif)$b < 0$, $a < 0$

The finite singular points are $O(0, 0)$, $A_i (i = 1, 2, 3, 4, 5, 6)$. $O(0, 0)$ is a nilpotent center. $A_1 = (0, 1/2\sqrt{-b})$ and $A_2 = (0, -1/2\sqrt{-b})$ are all centers. $A_i (i = 3, 4, 5, 6)$ are all saddles.

In the U_1 plane, there have two solutions $B_1(\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$ and $B_2(-\sqrt{\sqrt{(a^2-4bc)/(4b^2)} - a/(2b)}, 0)$. The linear part of system (5.1) is

$$\begin{pmatrix} -8ua - 16bu^3 & 0 \\ 0 & -2au - 4bu^3 \end{pmatrix}.$$

When $a < 0$, the singular points B_1 and B_2 are unstable and stable nodes, respectively.

The global phase portrait is shown in (l) of Fig. 1.

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References

[1] I.E. Colak, J. Llibre and C. Valls, *Hamiltonian linear type centers of linear plus cubic homogeneous polynomial vector fields*, Journal of Differential Equations, 2014, 257(5), 1623–1661.

[2] I.E. Colak, J. Llibre and C. Valls, *Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields*, Advances in Mathematics, 2014, 259, 655–687.

[3] I.E. Colak, J. Llibre and C. Valls, *Bifurcation diagrams for Hamiltonian linear type centers of linear plus cubic homogeneous polynomial vector fields*, Journal of Differential Equations, 2015, 258(3), 846–879.

[4] I.E. Colak, J. Llibre and C. Valls, *Bifurcation diagrams for Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields*, Journal of Differential Equations, 2017, 262(11), 5518–5533.

[5] M. Dukaric, J. Giné and J. Llibre, *Reversible nilpotent centers with cubic homogeneous nonlinearities*, Journal of Mathematical Analysis & Applications, 2016, 433(1), 305–319.

[6] F. Dias, J. Llibre and C. Valls, *Polynomial Hamiltonian systems of degree 3 with symmetric nilpotent centers*, Mathematics and Computers in Simulation, 2018, 144, 60–70.

[7] F. Dumortier, J. Llibre and J.C. ArtšŠs, *Qualitative Theory of Planar Differential Systems*, Universitext, Springer-Verlag, New York, 2006.

[8] B. García, J. Llibre and J.S. Pérez del Río, *Planar quasihomogeneous polynomial differential systems and their integrability*, Journal of Differential Equations, 2013, 255(10), 3185–3204.

[9] M. Han, et. al. *Polynomial Hamiltonian systems with a nilpotent critical point*, Advances in Space Research, 2010, 46(4), 521–525.

[10] H. Liang, J. Huang and Y. Zhao, *Classification of global phase portraits of planar quartic quasi-homogeneous polynomial differential systems*, Nonlinear Dynamics, 2014, 78(3), 1659–1681.