

# Weak Solutions of a Reaction Diffusion System with Superdiffusion and Its Optimal Control\*

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**Abstract** The existence and uniqueness of weak solutions to the 2-dimensional reaction diffusion system with superdiffusion and the optimal control of such model are investigated in this paper. Fractional function spaces, Galerkin approximation method and Gronwall inequality are used to obtain the existence and uniqueness of weak solutions. On this basis, an optimal control problem of such superdiffusive system is further considered by using the minimal sequence.

**Keywords** Weak solutions, optimal control, reaction-diffusion, Riesz operator.

**MSC(2010)** 35D30, 49J20.

## 1. Introduction

From a microscopic point of view, diffusion is usually described as the random motion of the individual particles among some media. Normal diffusion is often referred to as the Gaussian process, also known as the Brownian motion (Wiener process). In such process, the waiting time distribution and the jump length distribution between successive two jumps of particles must have finite moments. Due to the center limit theorem, it is characterized by the mean square displacement of a typical particle growing linearly with time  $\langle x^2(t) \rangle \propto t$ . Meanwhile, the anomalous diffusion are frequently observed, for example, in Refs. [6, 7, 10, 16–18] and references therein, in which the mean square displacement violates the linear relation with time and universally obeys the power-law relation, scaled as  $\langle x^2(t) \rangle \propto t^\gamma$ . When  $0 < \gamma < 1$ , it is called the subdiffusion, with the waiting time distribution having infinite moments, in this case the particle will wait for long times before next jumping, such phenomena could be found in porous media, polymers and gels, etc. When  $\gamma = 1$ , it just corresponds to the normal diffusion. If  $1 < \gamma < 2$ , it is said to be the superdiffusion, which is featured as the limiting result of Lévy flight, with the jumping length distribution having infinite moments. On such occasion, the particle will execute very long jumps. Such process could occur in the precesses of plasmas, turbulence, surface diffusion and motion of animals, etc.

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\*The authors were supported by the National Science Foundation of China (No.11571016, 61403115).

Since the anomalous diffusion has been theoretically speculated and experimentally found in nature, models, such as continuous time random walk, transport on fractals and fractional Brownian motion, have been established to manifest the anomalous processes, and differential equations with fractional order derivatives are presented to serve as a more appropriate mathematical tool to describe anomalous diffusion and the transmission dynamics of complex systems [2]. As opposed to the normal diffusion governed by the reaction-diffusion system with the standard Laplace operator  $\Delta$  in it, the superdiffusion process is described by the diffusive system with the Laplace operator replaced by the fractional order operator  $\nabla^\alpha$  ( $\alpha = \frac{2}{\gamma}$ ). So it is reasonable and necessary to take the behaviors of superdiffusive systems into consideration.

In recent years, many researchers consider the following reaction diffusion system with superdiffusion

$$\begin{cases} \frac{\partial u}{\partial t} - k_1 \nabla^\alpha u = f(u, v), & Q_T := (0, T] \times \Omega, \\ \frac{\partial v}{\partial t} - k_2 \nabla^\alpha v = g(u, v), & Q_T := (0, T] \times \Omega, \\ u(x, y, 0) = u_0, & v(x, y, 0) = v_0, \\ u(x, y, t) = 0, v(x, y, t) = 0, & \Sigma_T := [0, T] \times \mathbb{R}^2 \setminus \Omega, \end{cases} \quad (1.1)$$

where  $k_1$  and  $k_2$  are diffusion coefficients,  $1 < \alpha \leq 2$  and  $\Omega$  is a bounded open domain in  $\mathbb{R}^2$ . Reaction terms  $f$  and  $g$  can be expressed by distinct coupling reactions between  $u$  and  $v$ , which satisfy the Lipschitz conditions, i.e.,

$$|f(u_1, v_1) - f(u_2, v_2)| \leq L \|(u_1, v_1) - (u_2, v_2)\|, \quad (1.2)$$

$$|g(u_1, v_1) - g(u_2, v_2)| \leq L \|(u_1, v_1) - (u_2, v_2)\|, \quad (1.3)$$

for  $\forall (u_1, v_1), (u_2, v_2) \in \mathbb{R}^2$ , here  $L$  is a Lipschitz constant, with  $f(0, 0) = 0$ ,  $g(0, 0) = 0$ . The fractional operator  $\nabla^\alpha$  is a sequential Riesz fractional order operator in space [15], and could be given in [24] as follows:

$$\nabla^\alpha u = \frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\alpha u}{\partial |y|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)} [({}_x D_L^\alpha u + {}_x D_R^\alpha u) + ({}_y D_L^\alpha u + {}_y D_R^\alpha u)].$$

Here  $\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)} ({}_x D_L^\alpha u + {}_x D_R^\alpha u)$ ,  ${}_x D_L^\alpha$  and  ${}_x D_R^\alpha$  are defined, respectively

$$\begin{aligned} {}_x D_L^\alpha u &= \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-s)^{1-\alpha} u(s, y, t) ds, \\ {}_x D_R^\alpha u &= \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (s-x)^{1-\alpha} u(s, y, t) ds, \end{aligned}$$

where  $\alpha \in (1, 2)$ ,  $\Gamma(\cdot)$  is the Gamma function. The right Riemann-Liouville fractional derivative  $\frac{\partial^\alpha u}{\partial |y|^\alpha}$  can be defined similarly.

The existence and uniqueness of a solution are the basic theory of differential equations. However, research on the existence and uniqueness of the variational solution to the evolution equations with superdiffusion was delayed as a result of difficulties caused by fractional operators. For example, the fractional operator is non-local and the adjoint of a fractional differential operator is not the negative of itself. But, Ervin and Roop [3, 4, 19, 20] introduced fractional derivative spaces and

fractional Sobolev spaces to overcome these difficulties. Therefore, some progress has been made on the existence and uniqueness of solutions to the variational problem of fractional diffusion equations. In [1, 11, 25], the authors studied the existence and uniqueness of solutions to the variational problems of one-/two- dimensional Riesz fractional diffusion equations on bounded spaces by Galerkin approximation method. Meanwhile, the nonlinear terms of those equations are  $f(t, x)$  or  $f(t, x, y)$ . As we know, the anomalous diffusion is almost universal [18] and anomalous interaction within systems frequently occurs in the real world, such as in the predator-prey model [12, 13] and in inhibitor-activator system [14, 27], and so on. Note that many practical models take the form of (1). However, few results about the existence and uniqueness of solutions to system (1.1) have been reported. That motivates us to investigate system (1.1). To this end, fractional functional spaces and Galerkin approximation method will be formulated to deal with the 2-dimensional Riesz fractional operator. Therefore, it is desirable to study weak solution to the reaction diffusion system with superdiffusion.

Control theory has been applied in some fields such as engineering, ecology and computers for normal diffusion systems [5, 9] in recently years. In addition, there are some researches on feedback control and optimal control for subdiffusion systems [21, 22, 26]. In [21], Wang etc investigated optimal feedback controls of a system governed by semilinear fractional evolution equations via a compact semigroup in Banach spaces. In [22], the authors were concerned with feedback control systems governed by fractional impulsive evolution equations involving Riemann-Liouville derivatives in reflexive Banach spaces. In [26], Zhou and Peng gave a sufficient condition of optimal control pairs for the Navier-Stokes equations with the time-fractional derivative. However, there is few relevant results as to the study on the optimal control problem of superdiffusion systems. Hence, it motivates us to investigate the optimal control of the following system.

$$\begin{cases} \frac{\partial u}{\partial t} - k_1 \nabla^\alpha u = f(u, v), & Q_T := (0, T] \times \Omega, \\ \frac{\partial v}{\partial t} - k_2 \nabla^\alpha v = g(u, v) + \omega, & Q_T := (0, T] \times \Omega, \\ u(x, y, 0) = u_0, & v(x, y, 0) = v_0, \\ u(x, y, t) = 0, v(x, y, t) = 0, & \Sigma_T := [0, T] \times \mathbb{R}^2 \setminus \Omega, \end{cases} \quad (1.4)$$

where  $\omega \in L^\infty(Q_T)$ . The admissible control set is defined as

$$W = \{\omega(x, y, t) \in L^\infty(Q_T), -1 \leq \omega(x, y, t) \leq 1 \text{ a.e. in } Q_T\}.$$

Our ultimate goal is to find a control function  $\omega$  in  $W$  such that

$$J(U, \omega) = - \int_{Q_T} (l_1 u + l_2 v)(x, y, t) d\Omega dt - \int_{\Omega} (\delta_1 u + \delta_2 v)(x, y, T) d\Omega \quad (1.5)$$

has a minimum value at the optimal pair  $(U^*, \omega^*)$ , where  $U^* = (u^*, v^*)$  and  $l_i, \delta_i, i = 1, 2$  are all weights.

The remainder of this paper is organized as follows. Some function spaces and lemmas are introduced in Section 2. Existence and uniqueness of the weak solutions to system (1.1) will be investigated in Section 3. In Section 4, the existence of optimal pair to control system (1.4) is investigated.

## 2. Preliminaries

Some fractional derivative spaces, fractional Sobolev spaces and lemmas were introduced by Ervin and Roop in  $\mathbb{R}^2$  [3, 4]. Liu etc [23, 25] generalized these results in  $\Omega \subset \mathbb{R}^2$ . These spaces and lemmas will be recalled and used for later analysis in this paper. Throughout the paper,  $C_1, C_2, C_3, C_4$  and  $C$  are positive constants independent of  $u$  and  $v$ .

**Definition 2.1** (Definition 2.1, [25]). Assume  $\alpha > 0$ , define the seminorm of left fractional derivative space

$$|u|_{J_L^\alpha(\Omega)} = (\|_x D_L^\alpha u\|_{L^2(\Omega)}^2 + \|_y D_L^\alpha u\|_{L^2(\Omega)}^2)^{\frac{1}{2}},$$

and the norm of left fractional derivative space  $\|u\|_{J_L^\alpha(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + |u|_{J_L^\alpha(\Omega)}^2)^{\frac{1}{2}}$ , here  $J_L^\alpha(\Omega)$  ( $J_{L,0}^\alpha(\Omega)$ ) as the closure of  $C^\infty(\Omega)$  ( $C_0^\infty(\Omega)$ ) with respect to  $\|\cdot\|_{J_L^\alpha(\Omega)}$ , and  $C_0^\infty(\Omega)$  is the space of smooth functions with compact support in  $\Omega$ .

The right fractional derivative space  $|u|_{J_R^\alpha(\Omega)}$  and  $\|u\|_{J_R^\alpha(\Omega)}$  can be defined similarly.

**Definition 2.2** (Definition 2.3, [25]). Assume  $\alpha > 0$ ,  $\alpha \neq n - \frac{1}{2}, n \in N$ , define the seminorm of symmetric fractional derivative space

$$|u|_{J_S^\alpha(\Omega)} = (|(\|_x D_L^\alpha u, {}_x D_R^\alpha u\|_{L^2(\Omega)})| + |(\|_y D_L^\alpha u, {}_y D_R^\alpha u\|_{L^2(\Omega)})|)^{\frac{1}{2}},$$

and the norm of symmetric fractional derivative space  $\|u\|_{J_S^\alpha(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + |u|_{J_S^\alpha(\Omega)}^2)^{\frac{1}{2}}$ .

**Definition 2.3** (Definition 2.4, [25]). Let  $\alpha > 0$ , define the seminorm of fractional Sobolve space

$$|u|_{H^\alpha(\Omega)} = \| |\omega|^\alpha \mathcal{F}(\hat{u})(\omega) \|_{L^2(\mathbb{R}^2)}$$

and the norm of fractional Sobolve space  $\|u\|_{H^\alpha(\Omega)} = (\|u\|_{L^2(\Omega)}^2 + |u|_{H^\alpha(\Omega)}^2)^{\frac{1}{2}}$ , here  $\mathcal{F}(\hat{u})(\omega)$  is the Fourier transformation of function  $\hat{u}$ ,  $H^\alpha(\Omega)$  ( $H_0^\alpha(\Omega)$ ) as the closure of  $C^\infty(\Omega)$  ( $C_0^\infty(\Omega)$ ) with respect to  $\|\cdot\|_{H^\alpha(\Omega)}$ .

**Lemma 2.1** (Fractional Poincaré-Friedrichs inequality, Lemma 2.5, [23]). *If  $u \in H_0^\alpha(\Omega)$  and  $0 < s < \alpha$ , then*

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\leq C_1 \|_x D_L^s u\|_{L^2(\Omega)} \leq C_2 \|_x D_L^\alpha u\|_{L^2(\Omega)}, \\ \|u\|_{L^2(\Omega)} &\leq C_3 \|_y D_L^s u\|_{L^2(\Omega)} \leq C_4 \|_y D_L^\alpha u\|_{L^2(\Omega)}, \end{aligned}$$

**Lemma 2.2** (Lemma 2.6, [25]). *If  $\alpha > 0$ ,  $u \in J_{L,0}^\alpha(\Omega) \cap J_{R,0}^\alpha(\Omega)$ , then*

$$\begin{aligned} ({}_x D_L^\alpha u(x, y), {}_x D_R^\alpha u(x, y))_{L^2(\Omega)} &= \cos(\alpha\pi) \|_{-\infty} D_x^\alpha \hat{u} \|_{L^2(\Omega)}^2 = \cos(\alpha\pi) \|_x D_{+\infty}^\alpha \hat{u} \|_{L^2(\Omega)}^2, \\ ({}_y D_L^\alpha u(x, y), {}_y D_R^\alpha u(x, y))_{L^2(\Omega)} &= \cos(\alpha\pi) \|_{-\infty} D_y^\alpha \hat{u} \|_{L^2(\Omega)}^2 = \cos(\alpha\pi) \|_y D_{+\infty}^\alpha \hat{u} \|_{L^2(\Omega)}^2, \end{aligned}$$

here  $\hat{u}$  is the extension of  $u$  by zero outside  $\Omega$ .

**Lemma 2.3** (Lemma 2.7, [25]). *Assume  $u \in J_{L,0}^\alpha(\Omega) \cap J_{R,0}^\alpha(\Omega) \cap H_0^\alpha(\Omega)$ ,  $\alpha \neq n - \frac{1}{2}, n \in N$ , then*

$$C_1 |u|_{H^\alpha(\Omega)} \leq \max \left\{ |u|_{J_L^\alpha(\Omega)}, |u|_{J_R^\alpha(\Omega)} \right\} \leq C_2 |u|_{H^\alpha(\Omega)}.$$

**Remark 2.1.** If  $\alpha > 0, \alpha \neq n - \frac{1}{2}, n \in N$ , the semi-norms and norms of spaces  $J_{L,0}^\alpha(\Omega), J_{R,0}^\alpha(\Omega), J_{S,0}^\alpha(\Omega)$  and  $H_0^\alpha(\Omega)$  are equivalent.

**Lemma 2.4** (Lemma 2.8, [25]). *Assume  $u, v \in H_0^{\alpha/2}(\Omega) \cap H_0^\alpha(\Omega)$ ,  $\alpha \in (1, 2)$ , then*

$$({}_x D_L^\alpha u, v) = ({}_x D_L^{\alpha/2} u, {}_x D_R^{\alpha/2} v), \quad ({}_x D_R^\alpha u, v) = ({}_x D_R^{\alpha/2} u, {}_x D_L^{\alpha/2} v).$$

### 3. Main results

In this section, we will consider the existence and uniqueness of weak solutions of system (1.1).

The bilinear operator  $B(u, v)$  is defined as follows:

$$\begin{aligned} B(u, v) &= - \left( \frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\alpha u}{\partial |y|^\alpha}, v \right) \\ &= \frac{C}{2\cos\alpha\pi} [({}_x D_L^{\alpha/2} u, {}_x D_R^{\alpha/2} v) + ({}_x D_R^{\alpha/2} u, {}_x D_L^{\alpha/2} v) \\ &\quad + ({}_y D_L^{\alpha/2} u, {}_y D_R^{\alpha/2} v) + ({}_y D_R^{\alpha/2} u, {}_y D_L^{\alpha/2} v)]. \end{aligned}$$

From Theorem 1 in [24], we know that  $|B(u, v)| \leq C \|u\|_{H^{\frac{\alpha}{2}}(\Omega)} \|v\|_{H^{\frac{\alpha}{2}}(\Omega)}$ , i.e.,  $B(\cdot, \cdot)$  is continuous on  $H_0^{\frac{\alpha}{2}}(\Omega) \times H_0^{\frac{\alpha}{2}}(\Omega)$ .  $B(u, u) \geq \frac{C}{|\cos\alpha\pi|} \|u\|_{H^{\frac{\alpha}{2}}(\Omega)}^2$ , i.e.,  $B(\cdot, \cdot)$  is coercive over  $H_0^{\frac{\alpha}{2}}(\Omega)$ .

**Theorem 3.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  with smooth boundary  $\partial\Omega$  and the conditions (1.2) and (1.3) are satisfied. If  $u_0, v_0 \in L^2(\Omega)$ , then there exists a unique weak solution  $(u, v)$  of system (1.1).*

**Proof.** Assume that  $\{\phi_j\}_{j \geq 1}$  is a complete orthogonal basis in  $V_0 = H_0^{\frac{\alpha}{2}}(\Omega)$  and let  $V^m = \text{span}\{\phi_1, \dots, \phi_m\}$  be a finite dimensional subspace of  $H^{\frac{\alpha}{2}}(\Omega)$ .

The approximate problem will be considered. For each  $t \in [0, T]$ , we want to find  $u^m, v^m \in V^m$  such that

$$\begin{cases} \frac{d}{dt}(u^m(t), \phi_j) + k_1 B(u^m(t), \phi_j) = (f, \phi_j), \\ \frac{d}{dt}(v^m(t), \phi_j) + k_2 B(v^m(t), \phi_j) = (g, \phi_j), \end{cases} \quad (3.1)$$

with the initial conditions

$$u^m(0) = u_0^m = P^m(u_0) = \sum_{j=1}^m \rho_j \phi_j, \quad v^m(0) = v_0^m = P^m(v_0) = \sum_{j=1}^m \varsigma_j \phi_j,$$

where  $P^m$  is the orthogonal projection in  $L^2(\Omega)$  on  $V^m$ . The orthogonality of  $\{\phi_j\}_{j \geq 1}$  leads to  $\rho_j = \frac{(u_0, \phi_j)}{\phi_j \cdot \phi_j}$ , and that of  $\{\phi_j\}_{j \geq 1}$  leads to  $\varsigma_j = \frac{(v_0, \phi_j)}{\phi_j \cdot \phi_j}$ ,  $j = 1, 2, \dots, m$ .

Since  $u^m, v^m \in V^m$ , let  $u^m = \sum_{j=1}^m d_j^m \phi_j$  and  $v^m = \sum_{j=1}^m c_j^m \phi_j$ ,  $\{d_j^m\}_{j=1}^m$  and  $\{c_j^m\}_{j=1}^m$  are unknown and to be determined later.

Substituting  $u^m, v^m$  into (3.1), then linear ordinary differential equations of system (3.1) will be given as follows:

$$\begin{cases} \frac{d}{dt} d_j^m(\phi_j, \phi_j) + k_1 \sum_{i=1}^m B(\phi_i, \phi_j) d_i^m = (f, \phi_j), \\ \frac{d}{dt} c_j^m(\phi_j, \phi_j) + k_2 \sum_{i=1}^m B(\phi_i, \phi_j) c_i^m = (g, \phi_j), j = 1, 2, \dots, m. \end{cases}$$

Then from the standard theory of ODEs, there exists a unique solution vector  $\{d_j^m\}_{j=1}^m, \{c_j^m\}_{j=1}^m \in H^1(0, T)$ . As a result,  $u^m \in H^1(0, T; V_0)$ . Choose test function  $u^m, v^m$  in (3.1), respectively, then

$$\left(\frac{d}{dt} u^m, u^m\right) + k_1 B(u^m, u^m) = (f(u^m, v^m), u^m), \tag{3.2}$$

$$\left(\frac{d}{dt} v^m, v^m\right) + k_2 B(v^m, v^m) = (g(u^m, v^m), v^m). \tag{3.3}$$

Since  $B(u, u) \geq \frac{C}{|\cos\alpha\pi|} \|u\|_{H^{\alpha/2}(\Omega)}^2$ , from the Hölder inequality and the Cauchy inequality, then

$$\begin{aligned} |(f(u^m, v^m), u^m)| &\leq \int_{\Omega} |(u^m, v^m)| |u^m| d\Omega \\ &\leq \frac{1}{2} (\|u^m\|_{L^2(\Omega)}^2 + \|v^m\|_{L^2(\Omega)}^2) + \frac{1}{2} \|u^m\|_{L^2(\Omega)}^2, \\ |(g(u^m, v^m), u^m)| &\leq \int_{\Omega} |(u^m, v^m)| |u^m| d\Omega \\ &\leq \frac{1}{2} (\|u^m\|_{L^2(\Omega)}^2 + \|v^m\|_{L^2(\Omega)}^2) + \frac{1}{2} \|u^m\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that  $(u_t^m, u^m) = \int_{\Omega} u_t^m u^m d\Omega = \frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^m)^2 d\Omega$ . Therefore, there exist some constants  $C_1, C_2 > 0$ , which are independent of  $m$ , such that

$$\begin{aligned} \frac{d}{dt} \|u^m\|_{L^2(\Omega)}^2 + C \|u^m\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 &\leq C_1 \|u^m\|_{L^2(\Omega)}^2 + C_2 \|v^m\|_{L^2(\Omega)}^2, \\ \frac{d}{dt} \|v^m\|_{L^2(\Omega)}^2 + \bar{C} \|v^m\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 &\leq C_1 \|u^m\|_{L^2(\Omega)}^2 + C_2 \|v^m\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.4}$$

Adding inequality (3.4), then one has

$$\begin{aligned} \frac{d}{dt} (\|u^m\|_{L^2(\Omega)}^2 + \|v^m\|_{L^2(\Omega)}^2) + C (\|u^m\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 + \|v^m\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2) \\ \leq C_3 (\|u^m\|_{L^2(\Omega)}^2 + \|v^m\|_{L^2(\Omega)}^2), \end{aligned} \tag{3.5}$$

which implies that

$$\|u^m\|_{L^2(\Omega)}^2 + \|v^m\|_{L^2(\Omega)}^2 \leq e^{C_3 t} (\|u^m(0)\|_{L^2(\Omega)}^2 + \|v^m(0)\|_{L^2(\Omega)}^2). \tag{3.6}$$

Since

$$\|u^m(0)\|_{L^2(\Omega)}^2 = \left\| \sum_{k=1}^m (u_0, \phi_k) \phi_k \right\|_{L^2(\Omega)}^2 \leq \|u_0\|_{L^2(\Omega)}^2 \|\phi_k\|_{L^2(\Omega)}^4 = \|u_0\|_{L^2(\Omega)}^2$$

and

$$\|v^m(0)\|_{L^2(\Omega)}^2 = \left\| \sum_{k=1}^m (v_0, \phi_k) \phi_k \right\|_{L^2(\Omega)}^2 \leq \|v_0\|_{L^2(\Omega)}^2 \|\phi_k\|_{L^2(\Omega)}^4 = \|v_0\|_{L^2(\Omega)}^2,$$

there exists a constant  $C_4 > 0$ , which does not depend on  $m$ , such that

$$\max_{0 \leq t \leq T} \left( \|u^m(t)\|_{L^2(\Omega)} + \|v^m(t)\|_{L^2(\Omega)} \right) \leq C_4 \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right). \quad (3.7)$$

From inequality (3.5), then it follows that

$$\begin{aligned} & C \int_0^T \left( \|u^m(t)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 + \|v^m(t)\|_{H_0^{\frac{\alpha}{2}}(\Omega)}^2 \right) dt \\ & \leq C_3 \int_0^T \left( \|u^m(t)\|_{L^2(\Omega)}^2 + \|v^m(t)\|_{L^2(\Omega)}^2 \right) dt + \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.8)$$

From the above analysis, then we have

$$\|u^m(t)\|_{L^2(0,T;H_0^{\frac{\alpha}{2}}(\Omega))}^2 + \|v^m(t)\|_{L^2(0,T;H_0^{\frac{\alpha}{2}}(\Omega))}^2 \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right). \quad (3.9)$$

By the similar way, one can obtain that

$$\int_0^T \left( \left\| \frac{du^m}{dt} \right\|_{H^{-\frac{\alpha}{2}}}^2 + \left\| \frac{dv^m}{dt} \right\|_{H^{-\frac{\alpha}{2}}}^2 \right) dt \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right), \quad (3.10)$$

where  $H^{-\frac{\alpha}{2}}$  is the dual space of  $H_0^{\frac{\alpha}{2}}$ . Based on the above analysis, the sequences  $\{u^m\}$  and  $\{v^m\}$  are uniformly bounded in  $L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega))$ . Consequently, there exist two subsequences  $\{u^{m_l}\}_{l=1}^{\infty}$ ,  $\{v^{m_l}\}_{l=1}^{\infty}$  and two functions  $u, v$ , which satisfy  $u, v \in L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega))$  and  $u', v' \in L^2(0, T; H_0^{-\frac{\alpha}{2}}(\Omega))$ , such that

$$u^{m_l} \rightharpoonup u \text{ and } v^{m_l} \rightharpoonup v \text{ weakly in } L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega)), \text{ as } l \rightarrow \infty, \quad (3.11)$$

$$\frac{du^{m_l}}{dt} \rightharpoonup u' \text{ and } \frac{dv^{m_l}}{dt} \rightharpoonup v' \text{ weakly in } L^2(0, T; H_0^{-\frac{\alpha}{2}}(\Omega)), \text{ as } l \rightarrow \infty. \quad (3.12)$$

For fixed  $N$ , choose two functions  $\tilde{u}, \tilde{v} \in C^1(0, T; H_0^{\frac{\alpha}{2}}(\Omega))$ , which have the following expressions

$$\tilde{u}(t) = \sum_{k=1}^N d^k(t) \phi_k, \quad \tilde{v}(t) = \sum_{k=1}^N c^k(t) \phi_k, \quad (3.13)$$

where the functions  $\{d^k\}_{k=1}^N$  and  $\{c^k\}_{k=1}^N$  are given. Multiplying the first equation of (3.1) by  $d^k(t)$  and the second equation of (3.1) by  $c^k(t)$ , summing  $k$  from 1 to  $N$  and integrating with respect to the time variable  $t$ , then one gets

$$\begin{aligned} & \int_0^t \left( \left( \frac{du^m}{dt}, \tilde{u} \right) + B(u^m, \tilde{u}) \right) dt = \int_0^t (f(u^m, v^m), \tilde{u}) dt, \\ & \int_0^t \left( \left( \frac{dv^m}{dt}, \tilde{v} \right) + B(v^m, \tilde{v}) \right) dt = \int_0^t (g(u^m, v^m), \tilde{v}) dt. \end{aligned} \quad (3.14)$$

Let  $m = m_l$  and  $l \rightarrow \infty$ , in view of formulas (3.11), (3.12), then (3.14) becomes

$$\begin{aligned} & \int_0^t \left( \left( \frac{du}{dt}, \tilde{u} \right) + B(u, \tilde{u}) \right) dt = \int_0^t (f(u, v), \tilde{u}) dt, \\ & \int_0^t \left( \left( \frac{dv}{dt}, \tilde{v} \right) + B(v, \tilde{v}) \right) dt = \int_0^t (g(u, v), \tilde{v}) dt. \end{aligned} \quad (3.15)$$

Since the functions of (3.13) are dense in  $L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega))$ , then the formula (3.15) holds for any  $\tilde{u}, \tilde{v} \in L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega))$ .

Next, we need to prove  $u(0) = u_0, v(0) = v_0$ . Multiplying system (1.1) by  $\psi(t) \in C^1([0, T]; V_0)$  with  $\psi(T) = 0$ , then we could have

$$-\int_0^T (u, \psi') dt + k_1 \int_0^T B(u, \psi) dt = \int_0^T (f(u, v), \psi) dt + (u(0), \psi(0)), \quad (3.16)$$

$$-\int_0^T (v, \psi') dt + k_2 \int_0^T B(v, \psi) dt = \int_0^T (g(u, v), \psi) dt + (v(0), \psi(0)). \quad (3.17)$$

Through integration by parts for (3.14), then we have

$$-\int_0^T (u^m, \psi') dt + k_1 \int_0^T B(u^m, \psi) dt = \int_0^T (f(u^m, v^m), \psi) dt + (u^m(0), \psi(0)), \quad (3.18)$$

$$-\int_0^T (v^m, \psi') dt + k_2 \int_0^T B(v^m, \psi) dt = \int_0^T (g(u^m, v^m), \psi) dt + (v^m(0), \psi(0)). \quad (3.19)$$

Since  $u^m(0)$  and  $v^m(0)$  converge to  $u_0$  and  $v_0$  in  $L^2(\Omega)$ , respectively, when  $m \rightarrow \infty$ , then equations (3.18) and (3.19) become

$$-\int_0^T (u, \psi') dt + k_1 \int_0^T B(u, \psi) dt = \int_0^T (f(u, v), \psi) dt + (u_0, \psi(0)), \quad (3.20)$$

$$-\int_0^T (v, \psi') dt + k_2 \int_0^T B(v, \psi) dt = \int_0^T (g(u, v), \psi) dt + (v_0, \psi(0)). \quad (3.21)$$

By comparing (3.20) and (3.21) with (3.16) and (3.17), respectively, we get  $u(0) = u_0, v(0) = v_0$ . Therefore, from the above analysis, the weak solution of problem (1.1) exists in  $L^2(0, T; V_0)$ .

Finally, the uniqueness of solution of system (1.1) will be considered. Assume that  $(u_1(t), v_1(t))$  and  $(u_2(t), v_2(t))$  are two solutions of system (1.1) in  $V_0$  for a.e.  $t \in [0, T]$ . We denote  $u(t) := u_2(t) - u_1(t), v(t) := v_2(t) - v_1(t)$ , then

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} (|u(t)|^2 + |v(t)|^2) d\Omega + k_1 \int_{\Omega} |\nabla^{\frac{\alpha}{2}} u|^2 d\Omega + k_2 \int_{\Omega} |\nabla^{\frac{\alpha}{2}} v|^2 d\Omega = \\ & \int_{\Omega} (f(u_2, v_2) - f(u_1, v_1)) u d\Omega + \int_{\Omega} (g(u_2, v_2) - g(u_1, v_1)) v d\Omega. \end{aligned} \quad (3.22)$$

Using the Schwarz inequality and the Cauchy inequality together with assumptions (1.2)-(1.3), then we have

$$\int_{\Omega} (f(u_2, v_2) - f(u_1, v_1)) u d\Omega + \int_{\Omega} (g(u_2, v_2) - g(u_1, v_1)) v d\Omega \leq C \int_{\Omega} (|u(t)|^2 + |v(t)|^2) d\Omega.$$

Hence,

$$\begin{aligned} & \frac{d}{dt} (\|u(t)\|_{L^2(\Omega)}^2 + \|v(t)\|_{L^2(\Omega)}^2) + C(\|u\|_{H^{\frac{\alpha}{2}}(\Omega)} + \|v\|_{H^{\frac{\alpha}{2}}(\Omega)}) \\ & \leq C(\|u(t)\|_{L^2(\Omega)}^2 + \|v(t)\|_{L^2(\Omega)}^2). \end{aligned} \quad (3.23)$$



Then

$$\frac{d}{dt}(\|u(t)\|_{L^2(\Omega)}^2 + \|v(t)\|_{L^2(\Omega)}^2) \leq C(\|u(t)\|_{L^2(\Omega)}^2 + \|v(t)\|_{L^2(\Omega)}^2). \quad (3.24)$$

Since  $u(0) = u_2(0) - u_1(0) = 0$  and  $v(0) = v_2(0) - v_1(0) = 0$ , then inequality (3.24) implies that the solution is unique by the Gronwall inequality. This completes the proof.  $\square$

**Remark 3.1.** Theorem 2.2 in [8] is just the case  $\alpha = 2$  in our result.

**Remark 3.2.** It is found that  $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \in L^2(0, T; H^{-\frac{\alpha}{2}}(\Omega))$  from the proof of Theorem 3.1, so we obtain that  $u, v \in H^1(0, T; H^{-\frac{\alpha}{2}}(\Omega))$ , that means  $u, v \in L^2(0, T; H_0^{\frac{\alpha}{2}}(\Omega)) \cap H^1(0, T; H^{-\frac{\alpha}{2}}(\Omega))$ , that is to say,  $u, v \in C^0(0, T; L^2(\Omega))$ , see the Theorem 2 in [24].

## 4. Optimal control

In this section, the existence of the optimal pair to the control system (1.4) will be investigated.

**Theorem 4.1.** *Under the conditions of Theorem 3.1, the problem (1.5) admits an optimal pair  $(U^*, \omega^*)$  for the control system (1.4).*

**Proof.** From the results of Theorem 3.1, it is found that  $J(U, \omega)$  is bounded below. Hence, there exists a minimizing sequence  $\{\omega_m\}_{m \geq 1}$  and a constant  $\xi = \inf_{\omega \in W} J(U, \omega)$  such that

$$\xi = \lim_{m \rightarrow \infty} J(U_m, \omega_m) = \inf_{\omega \in W} J(U, \omega), \quad (4.1)$$

here  $U_m = (u_m, v_m)$  is the solution of the control system (1.4). From Remark 3.2 and the proof of Theorem 3.1, one can claim that  $U_m$  is uniformly bounded, i.e., there exists a positive constant, which is independent of  $m$ , such that

$$\|u_m\|_{L^\infty(Q_T)} + \|u_m\|_{L^2(0, T; H_0^{\frac{\alpha}{2}})} + \left\| \frac{\partial u_m}{\partial t} \right\|_{L^\infty(Q_T)} \leq C, \quad \forall t \in [0, T]. \quad (4.2)$$

When  $u_m$  changes to  $v_m$ , this inequality also holds. Moreover, from (4.2), one obtains the equicontinuity of the family  $\{u_m\}_{m \geq 1}$  and  $\{v_m\}_{m \geq 1}$ . Hence, by Ascoli-Arzela Theorem, there exist  $U^* = (u^*, v^*) \in C^0([0, T]; L^2(\Omega))$  and a subsequence of  $\{u_m\}_{m \geq 1}$  and  $\{v_m\}_{m \geq 1}$ , still denoted by itself, such that

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \|u_m - u^*\|_{L^2(\Omega)} = 0, \quad (4.3)$$

$$\lim_{m \rightarrow \infty} \sup_{t \in [0, T]} \|v_m - v^*\|_{L^2(\Omega)} = 0. \quad (4.4)$$

Therefore, one can prove that  $U^* = (u^*, v^*)$  is an optimal pair of control system (1.4) by letting  $m \rightarrow \infty$ . Further, since  $\omega_m$  is bounded in  $L^2(Q_T)$ , there exist a function  $\omega^*$  and a subsequence of  $\{\omega_m\}_{m \geq 1}$ , still denoted by itself, such that

$$\omega_m \rightarrow \omega^*, \text{ weakly in } L^2(Q_T). \quad (4.5)$$

From the property of set  $W$  and (4.5), one has that  $\omega^* \in W$ .

In terms of the analysis in Section 3, it is not difficult to get that

$$\begin{aligned} \frac{\partial u_m}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t}, \quad \frac{\partial v_m}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}, \quad \text{weakly in } L^2([0, T]; L^2(\Omega)), \\ u_m &\rightharpoonup u^*, \quad v_m \rightharpoonup v^*, \quad \text{weakly in } L^2([0, T]; H_0^{\frac{\alpha}{2}}(\Omega)), \\ f(u_m, v_m) &\rightharpoonup f(u^*, v^*), \quad g(u_m, v_m) \rightharpoonup g(u^*, v^*) \quad \text{weakly in } L^2([0, T]; L^2(\Omega)). \end{aligned}$$

In the following, the convergence of bilinear operator  $B(u, v)$  will be considered. From the continuity of  $B(u, v)$ ,  $\|u_m\|_V^2$  and  $\|v_m\|_V^2$  are uniformly integrable, then

$$\int_0^t |B(u_m, v_m)|_{V'} ds \leq C \int_0^t \|u_m\|_V \|v_m\|_V ds \leq C \|u_m\|_{L^2([0, T]; V)} \|v_m\|_{L^2([0, T]; V)} \leq \infty, \quad (4.6)$$

which implies that there exists  $\theta$  such that

$$B(u_m, v_m) \rightharpoonup \theta \quad \text{weakly in } L^2([0, T]; V'). \quad (4.7)$$

Moreover,

$$\begin{aligned} |(B(u_m, v_m) - B(u^*, v^*), \zeta)| &\leq |(B(u_m - u^*, v_m), \zeta)| + |(B(u^*, v_m - v^*), \zeta)| \\ &\leq C(\|u_m - u^*\|_V \|v_m\|_V + \|v_m - v^*\|_V \|u^*\|_V) \|\zeta\|_{V'}, \end{aligned} \quad (4.8)$$

for  $\forall \zeta \in V$ , then

$$B(u_m, v_m) \rightharpoonup B(u^*, v^*) \quad \text{weakly in } L^2([0, T]; V'). \quad (4.9)$$

Therefore,  $\theta = B(u^*, v^*)$  a.e.  $t \in [0, T]$ .

Let  $m \rightarrow \infty$ , then it is clear that  $(u_m, v_m, \omega_m) = (u^*, v^*, \omega^*)$  is a solution, i.e., it is an optimal pair to the control system (1.4). The proof is completed.  $\square$

## 5. Discussions

The existence, uniqueness and optimal control problem for the 2-dimensional reaction diffusion system with superdiffusion are investigated in this paper. Galerkin approximation method and Gronwall inequality are utilized to obtain the existence and uniqueness of weak solutions. On this basis, the existence of optimal control strategy is established by using the minimal sequence. For the control problem of superdiffusive systems, there are still interesting topics to work on, such as nonlocal controllability, first order necessary condition for optimal control, and so on. However, it is necessary to find out the the property of operator  $\nabla^\alpha$ , the existence of the adjoint operator of  $\nabla^\alpha$  and the expression of the adjoint operator, if it exists. These topics will be worthy of consideration in the future.

## Acknowledgments

This work was supported by the National Science Foundation of China (No.11571016, 61403115).

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