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Traveling Wave Solutions of a Fourth-order Generalized Dispersive and Dissipative Equation*

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Abstract In this paper, we consider a generalized nonlinear forth-order dispersive-dissipative equation with a nonlocal strong generic delay kernel, which describes wave propagation in generalized nonlinear dispersive, dissipation and quadratic diffusion media. By using geometric singular perturbation theory and Fredholm alternative theory, we get a locally invariant manifold and use fast-slow system to construct the desire heteroclinic orbit. Furthermore we construct a traveling wave solution for the nonlinear equation. Some known results in the literature are generalized.

Keywords Dispersive-dissipative equation, geometric singular perturbation, traveling waves, heteroclinic orbit.

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1. Introduction

In this paper, we are concerned with the existence of traveling wave solution for the generalized fourth-order dispersive and dissipative equation

$$\frac{\partial u}{\partial t} + \alpha u^n (f*u)\frac{\partial u}{\partial x} + \beta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^3 u}{\partial x^3} + s(\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})) + \delta\frac{\partial^4 u}{\partial x^4} = 0, \tag{1.1}$$

where $n \geq 1$, α, β, γ, s and δ are constant coefficients. u is a function of space x and time t, α is the nonlinear convective coefficient, β is the diffusion coefficient, γ is the dispersion coefficient, s is the backward quadratic diffusion coefficient and δ is the stable coefficient. Here, partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ indicate the corresponding partial differentiation with respect to spatial variable x and time variable t, respectively. $\frac{\partial^3 u}{\partial x^3}$, $\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})$ and $\frac{\partial^4 u}{\partial x^4}$ represent to dispersion effect term, backward quadratic diffusion term and the stable term, respectively. We take $f*u$ to be the following spatial-temporal convolution

$$(f*u)(x,t) = \int_{-\infty}^{t}\int_{-\infty}^{+\infty} f(x-y,t-s)u(y,s)dyds,$$

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here the function f satisfies the normalization conditions

$$f(t) \geq 0 \ for \ t \geq 0, \text{and} \ \int_{-\infty}^{t} \int_{+\infty}^{-\infty} f(x,t) dx dt = 1,$$

such that the kernel f doesn't affect the spatial-temporal uniform steady states.

The Eq.(1.1) describes wave propagation in generalized nonlinear dispersive, dissipation and quadratic diffusion media. It can be discovered in the context of Benard-Marangoni convection in shallow layers, thin liquid films, and so on [7]. Eq.(1.1) has many applications, for example, is governing evolution equation for the propagation of weak nonlinear waves in fluid-filled thick viscoelastic tubes for arterial blood flow. We point out that, if the parameters are chosen as different values, some famous equations can be derived from Eq.(1.1). For instance, if $n = 0$, $f * u = u$, $\beta = s = \delta = 0$, Eq.(1.1) becomes the Korteweg-de Vries (KdV, for short) equation [8]. As is known to that the KdV equation has been widely studied due to its significance in stratified internal wave, physical contexts, plasma physics and its applications in weakly nonlinear dispersive physical system [2,5,9,15,16]. When $n = 0$, $f * u = u$, $s = \varepsilon = 0$, Eq.(1.1) becomes the Burgers-KdV equation, which was first proposed the standard form by Feudel and Steudel [4] when they proved that the equation has no prolongation structure.

Mansour [11] considered a fourth order Burgers-KdV equation and proved the existence of traveling wave solutions. By using the dynamical systems theory, especially based upon the geometric singular perturbation theory and invariant manifold theorem, Mansour [12] constructed the traveling wave solutions of a nonlinear dispersive-dissipative equation. In many cases, differential equations with time delay can reflect the real natural phenomena. The delay is an significant factor that can not be ignored, which can make the steady state of the system change. Shang and Du [14] discussed the existence of traveling wave solutions to a nonlinear dispersive and dissipative equation

$$\frac{\partial u}{\partial t} + \alpha u^n (f * u) \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^3 u}{\partial x^3} + s \frac{\partial}{\partial x}(u \frac{\partial u}{\partial x}) = 0, \tag{1.2}$$

where $n \geq 1$, α, β, γ, and s are constant coefficients.

If the time delay disappears in Eq.(1.1), i.e., $f * u = u$, Mansour [12] found the existence of homolinic orbit of Eq.(1.1) by applying the method of the Melnikov function. We will get the existence of the heteroclinic orbit by using the invariant manifold on the phase plane. In the later part of the article, we also discuss the Eq.(1.1) with spatial-temporal delay, which describes the state that the system variables depend on the system at a certain time or in a certain historical period. If we choose $\delta = 0$, Eq.(1.1) becomes the Eq.(1.2) discussed by Shang and Du [14]. Our results agree well with the corresponding ones in [14]. In the case that Eq.(1.1) without delay, Shang and Du [14] obtained the existence of the heteroclnic orbit by constructing the triangular invariant set. However, our approach overcomes the difficulties that Eq.(1.1) adds the fourth order term by constructing the three pyramid invariant to get the heteroclinic orbit.

The remaining part of this article is organized as following. In section 2, we will construct the existence of traveling wave solutions of Eq.(1.1) without delay. In section 3, we will investigate Eq.(1.1) with a nonlocal delay. Using geometric singular perturbation theory [3,6] and Fredholm theorem, we get a locally invariant manifold and seek the heteroclinic orbit in this slow manifold. Furthermore we construct a traveling wave solution of Eq.(1.1). In section 4, we give a conclusion.

2. The existence of traveling waves for Eq.(1.1) without delay

In this section, we establish the existence result of traveling waves solutions for the Eq.(1.1) without delay, that is,

$$u_t + \alpha u^{n+1}u_x + \beta u_{xx} + \gamma u_{xxx} + s(uu_x)_x + \delta u_{xxxx} = 0, \tag{2.1}$$

discussed by the Mansour [12]. He established the existence of the homoclinic orbit to the equilibrium $(0,0)$ by using the Melnikov function method. We will construct the three pyramid invariant to obtain the heteroclinic orbit connecting two equilibria $E_1(0,0,0)$ and $E_2(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}},0,0)$.

In traveling wave transform, let $u(x,t) = \phi(z), z = x + ct$, then Eq.(2.1) can be reduced to the following traveling wave equation

$$c\phi' + \alpha\phi^{n+1}\phi' + \beta\phi'' + \gamma\phi''' + s(\phi\phi')' + \delta\phi'''' = 0, \tag{2.2}$$

where $' = \frac{d}{dz}$. By integrating once, without loss of generality, we set the integration constant equal to zero, then Eq.(2.3) becomes

$$c\phi + \frac{\alpha\phi^{n+2}}{n+2} + \beta\phi' + \gamma\phi'' + s\phi\phi' + \delta\phi''' = 0. \tag{2.3}$$

Eq.(2.3) can be rewrite as a three dimensional system consisting three first-order equations

$$\begin{cases} \phi' = \varphi_1, \\ \varphi_1' = \varphi_2, \\ \varphi_2' = \frac{1}{\delta}(-c\phi - \frac{\alpha}{n+2}\phi^{n+2} - \beta\varphi_1 - \gamma\varphi_2 - s\phi\varphi_1), \end{cases} \tag{2.4}$$

which has two equilibria $E_1(0,0,0)$ and $E_2(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}},0,0)$.

Theorem 2.1. *If $\alpha < 0$, $s < 0$ and $(\beta+1)^2 - 4\delta c > 0$ holds, then there exists a heteroclinic orbit connecting the critical point $E_1(0,0,0)$ and $E_2(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}},0,0)$ in the $(\phi,\varphi_1,\varphi_2)$ phase plane of system (2.4).*

Proof. By linearizing (2.4), we prove that the equilibrium E_1 is a saddle and the equilibrium E_2 is a unstable node. In order to show the existence of heteroclinic orbit connecting the two equilibria E_1 and E_2, we will prove that for a suitable value $\lambda > 0$, the three pyramid

$$M = \{(\phi,\varphi_1,\varphi_2) : 0 \le \phi \le \sqrt[n+1]{\frac{-(n+2)c}{\alpha}},\ 0 \le \varphi_1 \le \lambda\phi,\ 0 \le \varphi_2 \le \lambda\phi\},$$

is negative invariant (See the Figure 1).

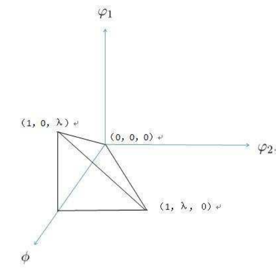

Figure 1. The three pyramid is negative invariant.

Let f is a vector which is defined by the right sides of system (2.4) and n is a normal vector on the surface of M. So we just prove $f \cdot n \leq 0$ on the four faces of the three pyramid and obtain

$$\begin{aligned}
f \cdot \overrightarrow{n} &= \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \frac{1}{\delta}(-c\phi - \frac{\alpha}{n+2}\phi^{n+2} - \beta\varphi_1 - \varphi_2 - s\phi\varphi_1) \end{pmatrix} \begin{pmatrix} \lambda, & -1, & -1 \end{pmatrix} \Big|_{(\phi,\lambda\phi,\lambda\phi)} \\
&= \lambda\varphi_1 - \varphi_2 + \frac{1}{\delta}(c\phi + \frac{\alpha}{n+2}\phi^{n+2} + \beta\varphi_1 + \varphi_2 + s\phi\varphi_1) \Big|_{(\phi,\lambda\phi,\lambda\phi)} \\
&= \lambda^2\phi - \lambda\phi + \frac{1}{\delta}(c\phi + \frac{\alpha}{n+2}\phi^{n+2} + \beta\lambda\phi + \lambda\phi + s\lambda\phi^2) \\
&= \lambda^2\phi + (-\phi + \frac{\beta\phi}{\delta} + \frac{\phi}{\delta} + \frac{s\phi^2}{\delta})\lambda + \frac{1}{\delta}(c\phi + \frac{\alpha}{n+2}\phi^{n+2}) \\
&\leq \lambda^2\phi + (-\phi + \frac{\beta\phi}{\delta} + \frac{\phi}{\delta})\lambda + \frac{1}{\delta}c\phi \\
&\leq (\lambda^2 + (\frac{\beta}{\delta} + \frac{1}{\delta})\lambda + \frac{c}{\delta})\phi.
\end{aligned} \tag{2.5}$$

According to the condition $(\beta+1)^2 - 4\delta c > 0$, we show

$$\lambda^2 + (\frac{\beta}{\delta} + \frac{1}{\delta})\lambda + \frac{c}{\delta} = 0$$

has two real positive roots with $0 < \lambda_1 \leq \lambda_2$. Under the conditions $0 < \lambda_1 \leq \lambda \leq \lambda_2$, $\delta > 0$ and $(\beta + 1) < 0$, one has

$$f \cdot \overrightarrow{n} \leq (\lambda^2 + (\frac{\beta}{\delta} + \frac{1}{\delta})\lambda + \frac{c}{\delta})\phi \leq 0.$$

Hence, one branch of the unstable manifold of E_2 always stay in the set M and connect the equilibrium point E_1. Then we get the existence of the desired heteroclinic orbit. □

3. The existence of traveling wave solutions to Eq.(1.1) with a nonlocal kernel

In this section, we will prove the existence of traveling waves for Eq.(1.1) with a nonlocal strong generic delay kernel [1, 10, 13, 14]

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{t}{\tau^2} e^{-\frac{t}{\tau}}, \tag{3.1}$$

where the $\tau > 0$ represents the average time delay.

We define that

$$v(x,t) = (f * u)(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{t-s}{\tau^2} e^{-\frac{t-s}{\tau}} u(y,s) dy ds, \tag{3.2}$$

then we get through the directly computation and obtain

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \frac{1}{\tau} W - \frac{1}{\tau} v, \tag{3.3}$$

where

$$W(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} \frac{1}{\tau} e^{-\frac{t-s}{\tau}} u(y,s) dy ds. \tag{3.4}$$

Then we can obtain

$$\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial x^2} + \frac{1}{\tau}(u - W). \tag{3.5}$$

Substituting (3.3) into (3.5), we obtain

$$\frac{\partial^2 v}{\partial t^2} = 2\frac{\partial^3 v}{\partial t \partial x^2} - \frac{\partial^4 v}{\partial x^4} + \frac{2}{\tau}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t}\right) + \frac{1}{\tau^2}(u - v). \tag{3.6}$$

Thus the equation (1.1) is reformulated as the following system

$$\begin{cases} u_t + \alpha u^n v u_x + \beta u_{xx} + \gamma u_{xxx} + s(u u_x)_x + \delta u_{xxxx} = 0, \\ v_{tt} = 2v_{txx} - v_{xxxx} + \frac{2}{\tau}(v_{xx} - v_t) + \frac{1}{\tau^2}(u - v). \end{cases} \tag{3.7}$$

Now the parameter τ is considered as the delay in the original system. Seeking the existence of traveling waves in Eq.(1.1) is changed into finding the existence of traveling waves in system (3.7).

However,we note that the Eq.(1.1) is an infinite dimensional dynamic system and the phase space for the traveling wave equations of system (3.7) is finite dimensional. So we tackle the traveling wave for the system (3.7) by employing the geometric singular perturbation theory. Since the delay $\tau \to 0$, $v(x,t) \to u(x,t)$. Then the system (3.7) becomes to the nonlocal model. The proof have already discussed in section 2, so we omit it here.

Let $u(x,t) = \phi(z)$, $v(x,t) = \varphi(z)$, $z = x + ct$. Since delay τ is not zero, then ϕ, φ satisfy the traveling wave system

$$\begin{cases} c\phi' + \alpha\phi^n \varphi \phi' + \beta\phi'' + \gamma\phi''' + s(\phi\phi')' + \delta\phi'''' = 0, \\ \varphi^4 - 2c\varphi''' + c^2\varphi'' - \frac{2}{\tau}(\varphi'' - c\varphi') - \frac{1}{\tau^2}(\phi - \varphi) = 0, \end{cases} \tag{3.8}$$

with the bounding value conditions satisfying

$$\lim_{z \to -\infty} (\phi(z), \varphi(z)) = (0, 0),$$

$$\lim_{z\to+\infty}(\phi(z),\varphi(z))=\left(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}},\ \sqrt[n+1]{\frac{-(n+2)c}{\alpha}}\right).$$

Integrating the first equation of the system (3.8), we obtain the differential systems

$$\begin{cases} c\phi+\dfrac{\alpha}{n+2}\phi^{n+1}\varphi+\beta\phi'+\gamma\phi''+s\phi\phi'+\delta\phi'''=0,\\ \varphi''''-2c\varphi'''+c^2\varphi''-\dfrac{2}{\tau}(\varphi''-c\varphi')-\dfrac{1}{\tau^2}(\phi-\varphi)=0. \end{cases} \tag{3.9}$$

Now we define some new variables $\phi'=\phi_1$, $\varphi'=\varphi_1$, $\varphi_1'=\varphi_2$, $\varphi_2'=\varphi_3$, then system (3.9) can be transformed into a seven-dimensional system

$$\begin{cases} \phi'=\phi_1,\\ \phi_1'=\phi_2,\\ \phi_2'=\dfrac{1}{\delta}(-c\phi-\dfrac{\alpha}{n+2}\phi^{n+1}\varphi-\beta\phi_1-\gamma\phi_2-s\phi\phi_1),\\ \varphi'=\varphi_1,\\ \varphi_1'=\varphi_2,\\ \varphi_2'=\varphi_3,\\ \varphi_3'=2c\varphi_3-c^2\varphi_2+\dfrac{2}{\tau}(\varphi_2-c\varphi_1)+\dfrac{1}{\tau^2}(\phi-\varphi). \end{cases} \tag{3.10}$$

We introduce the small parameter $\varepsilon=\sqrt{\tau}$ and redefine new variables

$$u_1=\phi,\ u_2=\phi_1,\ u_3=\phi_2,\ v_1=\varphi,\ v_2=\varepsilon\varphi_1,\ v_3=\varepsilon^2\varphi_2,\ v_4=\varepsilon^3\varphi_3,$$

thus the system (3.10) can be changed into a standard singularly perturbed problem

$$\begin{cases} u_1'=u_2,\\ u_2'=u_3,\\ u_3'=\dfrac{1}{\delta}(-cu_1-\dfrac{\alpha}{n+2}u_1^{n+1}v_1-\beta u_2-\gamma u_3-su_1u_2),\\ \varepsilon v_1'=v_2,\\ \varepsilon v_2'=v_3,\\ \varepsilon v_3'=v_4,\\ \varepsilon v_4'=2c\varepsilon v_4-c^2\varepsilon^2v_3+2(v_3-c\varepsilon v_2)+u_1-v_1. \end{cases} \tag{3.11}$$

If we choose $\varepsilon=0$, system (3.11) could be reduced into the third order differential equations (2.4). According to Theorem 2.1, we know that system (3.11) with $\varepsilon=0$ has traveling wave solutions. When $\varepsilon>0$ is a sufficiently small parameter, then it isn't a dynamic in R^7. This problem may be solved by using the transformation

$z = \varepsilon\eta$, and the system (3.11) translates into the following system

$$\begin{cases} \dot{u_1} = \varepsilon u_2, \\ \dot{u_2} = \varepsilon u_3, \\ \dot{u_3} = \dfrac{\varepsilon}{\delta}(-cu_1 - \dfrac{\alpha}{n+2} u_1^{n+1} v_1 - \beta u_2 - \gamma u_3 - s u_1 u_2), \\ \dot{v_1} = v_2, \\ \dot{v_2} = v_3, \\ \dot{v_3} = v_4, \\ \dot{v_4} = 2c\varepsilon v_4 - c^2\varepsilon^2 v_3 + 2(v_3 - c\varepsilon v_2) + u_1 - v_1, \end{cases} \tag{3.12}$$

where dots denote differentiation with respect to η. These systems (3.11) and (3.12) are equivalent when $\varepsilon > 0$ [6]. The different time scales cause two different limiting systems. Let $\varepsilon = 0$, then the flow of the slow system (3.11) is confined to the set

$$M_0 = \{(u_1, u_2, u_3, v_1, v_2, v_3, v_4) \in R^7 : v_2 = 0, v_3 = 0, v_4 = 0 \ and \ u_1 = v_1\},$$

which is a three-dimensional invariant manifold of (3.11) when $\varepsilon = 0$ and its dynamics are just determined by the first three equations only. Generally, system (3.11) is known as the slow system, since the time scale z is slow and the system (3.12) is called the fast system, since the time scale η is fast. M_0 is slow manifold.

If the manifold M_0 is normally hyperbolic, for efficiently small $\varepsilon > 0$, by applying the geometric singular perturbation theory of Fenichel [3], we obtain a three-dimensional invariant manifold M_ε of system (3.11) when $\tau > 0$, which is close to M_0 and deduces the existence of the slow manifold as well as the stable and unstable foliations. Therefore, we just need to research the flow of the slow system (3.11) restricted to M_ε and show that the three-dimensional reduced the existence of a heteroclinic orbit.

We call that M_0 is said to be a normally hyperbolic if the linearization of the fast system, restricted to M_0, has exactly $\dim M_0$ eigenvalues on the imaginary axis $R(\lambda) = 0$ [3, 6, 13]. It is easy to see that he linearized matrix of system (3.12) restricted to M_0 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 2 & 0 \end{pmatrix}.$$

The matrix has seven eigenvalues: $0,\ 0,\ 0,\ 1,\ 1,\ -1,\ -1$, we have the correct number of eigenvalues on the imaginary axis and the other eigenvalues are hyperbolic. According to definition, we could obtain the invariant manifold M_0 is normally hyperbolic.

We need the following results on the invariant manifolds, which is established by Fenichel [3]. We use a form of this theorem due to Jones [6].

Lemma 3.1 (geometric singular perturbation theorem, [6]). *For the system*

$$\begin{cases} x'(t) = f(x, y, \varepsilon), \\ y'(t) = \varepsilon g(x, y, \varepsilon), \end{cases} \tag{3.13}$$

where $' = \frac{d}{dt}$, $x \in R^n$, $y \in R^l$ *with* $n,\ l \geq 1$ *in general and* ε *is a real parameter.* f, g *are* C^∞ *on the set* $V \times I$ *where* $V \in R^{n+l}$ *and* I *is an open interval containing* 0. *If when* $\varepsilon = 0$, *the system has a compact, normally hyperbolic manifold of critical points* M_0, *which is contained in the set* $\{f(x, y, 0) = 0\}$. *Then for any* $0 < r < +\infty$, *if* $\varepsilon > 0$, *but sufficiently small, there exists a manifold* M_ε, *satisfying*

(I) which is locally invariant under the flow of (3.13);

(II) which is C^r *in* x, y *and* ε*;*

(III) $M_\varepsilon = \{(x, y) : x = h^\varepsilon(y)\}$ *for some* C^r *function* $h^\varepsilon(y)$ *and* y *in some compact* K*;*

(IV) there exist locally invariant stable and unstable manifolds $W^s(M_\varepsilon)$ *and* $W^u(M_\varepsilon)$ *that lie within* $o(\varepsilon)$, *and are diffeomorphic to* $W^s(M_0)$ *and* $W^u(M_0)$.

From geometric singular perturbation theorem, for sufficiently small $\varepsilon > 0$, we known that there exists a sub-manifold M_ε of the perturbed system (3.11), which can be written as

$$M_\varepsilon = \{(u_1, u_2, u_3, v_1, v_2, v_3, v_4) \in R^7 : v_1 = g(u_1, u_2, u_3, \varepsilon)+u_1, v_2 = h(u_1, u_2, u_3, \varepsilon),$$
$$v_3 = k(u_1, u_2, u_3, \varepsilon), v_4 = r(u_1, u_2, u_3, \varepsilon)\},$$

where g, h, k and r depend smoothly on ε, are to be determined and satisfy

$$g(u_1, u_2, u_3, 0) = h(u_1, u_2, u_3, 0) = k(u_1, u_2, u_3, 0) = r(u_1, u_2, u_3, 0) = 0. \tag{3.14}$$

Since the functions g, h, k and r are zero, when $\varepsilon = 0$, thus these functions could be expanded into the form of a Taylor series about the delay ε

$$\begin{aligned} g(u_1, u_2, u_3, \varepsilon) &= \varepsilon g_1 + \varepsilon^2 g_2 + \cdots, \\ h(u_1, u_2, u_3, \varepsilon) &= \varepsilon h_1 + \varepsilon^2 h_2 + \cdots, \\ k(u_1, u_2, u_3, \varepsilon) &= \varepsilon k_1 + \varepsilon^2 k_2 + \cdots, \\ r(u_1, u_2, u_3, \varepsilon) &= \varepsilon r_1 + \varepsilon^2 r_2 + \cdots. \end{aligned} \tag{3.15}$$

Substituting $v_1 = g + u_1$, $v_2 = h$, $v_3 = k$ and $v_4 = r$ into the slow system (3.11), we obtain

$$\begin{cases} \varepsilon[\frac{\partial g}{\partial u_1}u_2+\frac{\partial g}{\partial u_2}u_3+\frac{\partial g}{\partial u_3}\frac{1}{\delta}(-cu_1 - \frac{\alpha}{n+2}u_1^{n+1}(g + u_1)-\beta u_2-\gamma u_3 - su_1u_2)+u_2]=h, \\ \varepsilon[\frac{\partial h}{\partial u_1}u_2+\frac{\partial h}{\partial u_2}u_3+\frac{\partial h}{\partial u_3}\frac{1}{\delta}(-cu_1 - \frac{\alpha}{n+2}u_1^{n+1}(g + u_1)-\beta u_2-\gamma u_3 - su_1u_2)] = k, \\ \varepsilon[\frac{\partial k}{\partial u_1}u_2+\frac{\partial k}{\partial u_2}u_3+\frac{\partial k}{\partial u_3}\frac{1}{\delta}(-cu_1 - \frac{\alpha}{n+2}u_1^{n+1}(g + u_1)-\beta u_2-\gamma u_3 - su_1u_2)] = r, \\ \varepsilon[\frac{\partial r}{\partial u_1}u_2+\frac{\partial r}{\partial u_2}u_3+\frac{\partial r}{\partial u_3}\frac{1}{\delta}(-cu_1 - \frac{\alpha}{n+2}u_1^{n+1}(g + u_1)-\beta u_2-\gamma u_3 - su_1u_2)] \\ \qquad = 2c\varepsilon r - c^2\varepsilon^2 k + 2(k - c\varepsilon h) - g. \end{cases} \tag{3.16}$$

Substituting (3.15) into (3.16), comparing coefficients of the parameter ε and ε^2, one has

$$g_1(u_1,u_2,u_3)=0, g_2(u_1,u_2,u_3)=\frac{1}{\delta}(-cu_1-\frac{\alpha}{n+2}u_1^{n+2}-\beta u_2-\gamma u_3-su_1u_2),$$
$$h_1(u_1,u_2,u_3)=u_2, h_2(u_1,u_2,u_3)=0,$$
$$k_1(u_1,u_2,u_3)=0, k_2(u_1,u_2,u_3)=u_3,$$
$$r_1(u_1,u_2,u_3)=0, r_2(u_1,u_2,u_3)=\frac{1}{\delta}(-cu_1-\frac{\alpha}{n+2}u_1^{n+2}-\beta u_2-\gamma u_3-su_1u_2).$$

Hence, we obtain

$$\begin{aligned} g(u_1,u_2,u_3,\varepsilon)&=g_2\varepsilon^2+o(\varepsilon^3);\\ h(u_1,u_2,u_3,\varepsilon)&=u_2\varepsilon+o(\varepsilon^3);\\ k(u_1,u_2,u_3,\varepsilon)&=u_3\varepsilon^2+o(\varepsilon^3);\\ r(u_1,u_2,u_3,\varepsilon)&=r_2\varepsilon^2+o(\varepsilon^3). \end{aligned} \tag{3.17}$$

The slow system (3.11) which is restricted to M_ε is given as follows

$$\begin{cases} u_1'=u_2,\\ u_2'=u_3,\\ u_3'=\dfrac{1}{\delta}[-cu_1-\dfrac{\alpha}{n+2}u_1^{n+1}(g+u_1)-\beta u_2-\gamma u_3-su_1u_2], \end{cases} \tag{3.18}$$

here g is given by (3.17). It is easy to show that when $\varepsilon=0$, the system (3.18) reduces to the corresponding system (2.4) for the non-delay equation. For any sufficiently small $\varepsilon>0$, then system (3.17) exists two two equilibria $E_1(0,0,0)$ and $E_2(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}},0,0)$. In the following theorem, we will establish the existence result of a heteroclinic connection between these two critical points. Therefore the Eq. (1.1) exists a travelling wave solution connecting E_1 and E_2.

Theorem 3.1. *Suppose $\tau>0$ is an any sufficiently small constant, then there has the speed $c<\frac{\beta^2}{3\gamma}$, such that Eq.(1.1) with the strong kernel (3.1) exists a traveling wave, $u(x,t)=\phi(x+ct)$ which connects the two equilibria E_1 and E_2.*

Proof. We will obtain the existence of a heteroclinic connection between these two equilibria E_1 and E_2. By Theorem 2.1, for $\varepsilon=0$, we see that such a connection exists. For $\forall\varepsilon>0$, we set

$$u_1=u_0+\varepsilon^2\phi+\cdots, u_2=\widehat{u}_0+\varepsilon^2\phi_1+\cdots, u_3=\tilde{u}_0+\varepsilon^2\phi_2+\cdots$$

substituting into (3.18), furthermore comparing the coefficients of the parameter ε^2, then the differential system determining ϕ and φ is given

$$\frac{d}{dz}\begin{pmatrix}\phi(z)\\ \phi_1(z)\\ \phi_2(z)\end{pmatrix}+\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & -1\\ \frac{c}{\delta}+\frac{2\alpha}{\delta(n+2)}u_0^{n+1}+\frac{s\widehat{u}_0}{\delta} & \frac{\beta}{\delta} & \frac{\gamma+su_0}{\delta}\end{pmatrix}\begin{pmatrix}\phi\\ \phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}0\\ 0\\ \frac{-\frac{\alpha}{n+2}u_0^{n+1}}{\delta}G(u_0,\widehat{u}_0,\tilde{u}_0)\end{pmatrix}, \tag{3.19}$$

where

$$G(u_0,\widehat{u}_0,\tilde{u}_0)=\frac{1}{\delta}(-cu_0-\frac{\alpha}{n+2}u_0^{n+2}-\beta\widehat{u}_0-\gamma\tilde{u}_0-su_0\widehat{u}_0).$$

Then the system exists a solution satisfying $\phi(\pm\infty) = 0$, $\phi_1(\pm\infty) = 0$, and $\phi_2(\pm\infty) = 0$.

Let L^2 be the space of square integrable functions, with the following inner production

$$\int_{-\infty}^{+\infty} (x(z),\ y(z))dz,$$

$(\cdot\ ,\ \cdot)$ being the Euclidean inner product on R^2. In view of Fredholm theory, we know (3.19) has a solution if and only if

$$\int_{-\infty}^{+\infty} \left(x(z), \begin{pmatrix} 0 \\ 0 \\ \frac{-\alpha u_0^{n+1}}{\delta(n+2)} G(u_0, \widehat{u}_0, \tilde{u}_0) \end{pmatrix} \right) dz = 0,$$

for all functions $x(z)$ in the kernel of the adjoint of operator L defined by left side of (3.19). It is easy to see that the adjoint operator L^* is as follows

$$L^* = -\frac{d}{dz} + \begin{pmatrix} \frac{\beta}{\delta} & \frac{\gamma+su_0}{\delta} & 1 \\ -(\frac{c}{\delta} + \frac{2\alpha}{\delta(n+2)} u_0^{n+1} + \frac{s\hat{u}_0}{\delta}) & 0 & 0 \\ 0 & -(\frac{c}{\delta} + \frac{2\alpha}{\delta(n+2)} u_0^{n+1} + \frac{s\hat{u}_0}{\delta}) & 0 \end{pmatrix}. \quad (3.20)$$

By computing the $KerL^*$, we show that all $x(z)$ satisfies the following system

$$\frac{dx(z)}{dz} = \begin{pmatrix} \frac{\beta}{\delta} & \frac{\gamma+su_0}{\delta} & 1 \\ -(\frac{c}{\delta} + \frac{2\alpha}{\delta(n+2)} u_0^{n+1} + \frac{s\hat{u}_0}{\delta}) & 0 & 0 \\ 0 & -(\frac{c}{\delta} + \frac{2\alpha}{\delta(n+2)} u_0^{n+1} + \frac{s\hat{u}_0}{\delta}) & 0 \end{pmatrix} x(z). \quad (3.21)$$

Because the matrix is nonconstant, It is difficult to seek the general solution of system (3.21). But we only need to find the solutions satisfying $x(\pm\infty) = 0$. In fact, the only such solution is a zero solution. we recall that $u_0(z)$ is the solution of the unperturbed problem and although we have no explicit expression for it. when $z \to -\infty$, it tends to zero. In (3.21), we let $z \to -\infty$, then the matrix turns into a constant matrix, with the eigenvalues λ satisfying the following algebraic equation

$$\lambda^3 - \frac{\beta}{\delta}\lambda^2 + \frac{c\gamma}{\delta^2}\lambda + \frac{c^2}{\delta^2} = 0.$$

Since $c < \dfrac{\beta^2}{3\gamma}$, we show the eigenvalues are real and negative. Therefore, when $z \to -\infty$, the solution of system (3.21), other than the zero solution, have to be decreasing exponentially for small z. So the only solution satisfying $x(\pm\infty) = 0$ is the zero solution. Using the Fredholm orthogonality condition, the solutions of (3.20) exist which satisfy $\phi(\pm\infty) = 0$, $\phi_1(\pm\infty) = 0$ and $\phi_2(\pm\infty) = 0$. Therefore for sufficiently small $\varepsilon > 0$, there has a heteroclinic orbit of (3.18) connecting these two equilibria $E_1(0,0,0)$ and $E_2(\sqrt[n+1]{\dfrac{-(n+2)c}{\alpha}}, 0, 0)$. Furthermore, while $\tau > 0$ is sufficiently small, there exists a heteroclinic orbit connecting these two equilibria $(0,0,0,0)$ and $\left(\sqrt[n+1]{\frac{-(n+2)c}{\alpha}}, 0, \sqrt[n+1]{\frac{-(n+2)c}{\alpha}}, 0\right)$. Hence, Eq.(1.1) has a traveling wave solution connecting these two equilibria. □

Remark 3.1. If taking the local delay kernel into a weak kernel, i.e.

$$f_1(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \frac{1}{\tau} e^{-\frac{t}{\tau}},$$

the existence result of traveling waves could be similarly established.

Remark 3.2. If we take $f * u = u$ in Eq.(1.1), we find the corresponding equation is a special case of Eq.(1.1) without delay, which was discussed by Mansour in [12]. In fact, methods and results are both different from those in [12]. Mansour [12] obtained the homoclinic orbit due to the method of dynamic systems, specifically base upon the center manifold theorem and Melnikov function. However, we get the existence of the heteroclinic orbit by using the invariant manifold on the phase plane of (2.4) which is different from Masour [12].

Remark 3.3. If we choose $\delta = 0$, then Eq.(1.1) becomes a third order dispersive-dissipative equation, which was investigated by the Shang and Du in [14]. It is notable that our results agree well with the corresponding ones in [14]. In the case that Eq.(1.1) without delay, Shang and Du established the existence of the heteroclinic orbit by constructing the triangular invariant set. However, we establish the traveling waves by constructing the three pyramid invariant set and getting the desired heteroclinic orbit.

4. Conclusions

In this paper, we are concerned with a fourth-order generalized nonlinear dispersive and dissipative equation with strong generic delay kernel. We obtain the existence of travelling wave solutions due to the geometric singular perturbation theory. In fact, if we choose the kernel f as the weak generic delay, the method and approach in this paper is still applicable, we omit it. We will discuss the asymptotic behavior of traveling waves of Eq.(1.1) in the future. In the discussion of the asymptotic behavior, since the interact term $\phi\phi'$ appears in system (3.9), it is difficult to seek the general solutions of system (3.9).

References

[1] Z.J. Du, J. Li and X.W. Li, *The existence of solitary wave solutions of delayed Camassa-Holm via a geometric approach*, Journal of Functional Analysis, 2018, 275, 988–1007.

[2] L. Escauriaza, C.E. Kenig and G. Ponce, *On uniqueness properties of solutions of the k-generalized KdV equations*, Journal of Functional Analysis, 2007, 244, 504–535.

[3] N. Fenical, *Geometric singular perturbation theory for ordinary differential equations*, Journal of Differetial Equations, 1979, 31, 53–98.

[4] F. Feudel and H. Steudel, *Non-existence of prolongation structure for the Korteweg-de Vries-Burgers equation*, Physics Letters. A, 1985, 107, 5–8.

[5] S. Ghosh and N. Chakrabarti, *Nonlinear low-frequency electrostatic wave dynamics in a two-dimensional quantum plasma*, Annals of Physics, 2016, 371, 67–76.

[6] Christopher K.R.T. Jones, *Geoetric singular perturbation theory*, In Dynamical Systems Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1995, 1609, 44–118.

[7] I.P. Kliakhandler, A.V. Porubov, and M.G. Velarde, *Localized finite-amplitude disturbances and selection of solitary waves*, Physical Review E, 2000, 62, 4959–4962.

[8] D.J. Korteweg and G.de. Vries, *On the change of form of long waves advancing in a rectangular channel and on a new type of long stationary waves*, Philosophical Magazine Series 5, 1895, 39, 422–443.

[9] M.J. Lee and Y.D. Jung, *Bohm potential effect on the propagation of electrostatic surface wave in semi-bounded quantum plasmas*, Physics Letters. A, 2017, 281, 636–638.

[10] G.Y. Lv and M.X. Wang, *Existence, uniqueness and asymptotic behavior of traveling wave fronts for a vector disease model*, Nonlinear Analysis. Real World Applications, 2010, 11, 2035–2043.

[11] M.B.A. Mansour, *Traveling wave solutions of the Burgers-KdV equation with a fourth order term*, Reports on Mathematical Physics, 2009, 63, 153–161.

[12] M.B.A. Mansour, *A geometric construction of traveling waves in a generalized nonlinear dispersive-dissipative equation*, Journal of Geometry and Physics, 2013, 69, 116–122.

[13] S.G. Ruan and D.M. Xiao, *Stability of steady states and existence of travelling waves in a vector-disease model*, Proceedings of the Royal Society of Edinburgh. Section A, 2004, 134, 991–1011.

[14] X.H. Shang and Z.J. Du, *Existence of Traveling waves in a generalized nonlinear dispersive-dissipative equation*, MathematicalMethods in the Applied Sciences, 2016, 39, 3035–3042.

[15] T. Tao, *Scattering for the quartic generalized Korteweg-de Vries equation*, Journal of Differential Equations, 2007, 232, 623–651.

[16] K.G. Zhuang, Z.J. Du and X.J. Lin, *Solitary waves solutions of singularly perturbed higher-order KdV equation via geometric singular perturbation method*, Nonlinear Dynamics, 2015, 80, 629–635.